Joint eigenfunctions for the relativistic Calogero-Moser Hamiltonians of hyperbolic type

Martin Hallnäs[1] Simon Ruijsenaars[2]

[1]Dept. of Mathematical Sciences, Loughborough University, UK
[2]School of Mathematics, University of Leeds, UK

Background

The sine-Gordon equation

$$(\partial_x^2 - \partial_t^2)\phi = \sin\phi \tag{1}$$

defines a relativistically invariant field theory. Depending on how the field ϕ is interpreted, one can view (1) either as a classical nonlinear evolution equation or as an interacting relativistic quantum field theory.
A remarkable feature of both the classical and quantum sine-Gordon models is the presence of 'solitons'. In the quantum case this means that particle creation and annihilation is absent, in a collision the set of momenta is conserved, and the scattering operator for a N-particle collision factorizes as a product of all pair scattering operators.
Some thirty years ago this led one of us (S.R.) to ask the following question:

> *Do there exist Hamiltonian dynamics for N point particles that lead to the same factorized scattering?*

In the classical case this question has been answered in the affirmative. An important aim of the present work is to show that the answer is affirmative also in the quantum case.

A relativistic Calogero-Moser system

The relevant N-particle system is the so-called relativistic Calogero-Moser system of hyperbolic type. In the quantum case this system is given by $2N$ commuting Hamiltonians

$$H_{k,\delta} \equiv \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=k}} \prod_{\substack{m\in I\\ n\notin I}} f_{\delta,-}(x_m - x_n) \prod_{l\in I} \exp(-ia_{-\delta}\partial_{x_l}) \prod_{\substack{m\in I\\ n\notin I}} f_{\delta,+}(x_m - x_n),$$

where $k = 1,\dots,N$, $\delta = +,-$, and

$$f_{\delta,\pm}(z) = \left(\frac{\sinh \pi(z \pm ib)/a_\delta}{\sinh \pi z/a_\delta}\right)^{1/2}.$$

Physical picture: For $\delta = +$, there are two length scales, namely

$$a_+ \equiv 2\pi/\mu, \quad \text{(imaginary period/interaction length)},$$

and

$$a_- \equiv \hbar/mc, \quad \text{(shift step size/Compton wavelength)},$$

with $\hbar$ Planck's constant, m particle mass and c the speed of light; $b = a_+/2$ corresponds to the sine-Gordon model. (For $\delta = -$, the length scales are interchanged.)
Together with the boost $B = -m\sum_{j=1}^N x_j$, the time and space translation generators $mc^2[H_{1,+}(x) + H_{1,+}(-x)]$ and $mc[H_{1,+}(x) - H_{1,+}(-x)]$ form a representation of the Lie algebra of the Poincaré group.

Goals

We aim to obtain the following results:
- Construct modular invariant ($a_+ \leftrightarrow a_-$) joint eigenfunctions of the Hamiltonians $H_{k,\delta}$.
- Establish orthogonality and completeness of the eigenfunctions.
- Thus reinterpret the commuting Hamiltonians $H_{k,\delta}$ as commuting self-adjoint operators on the Hilbert space

$$L^2(F_N, dx), \quad F_N = \{-\infty < x_N < \cdots < x_1 < \infty\}.$$

- Prove the S-operator factorizes (soliton scattering).

For $b = a_+/2$, we expect that this will reproduce the scattering in the sine-Gordon model.

Kernel functions

To construct joint eigenfunctions our main tool is so-called kernel functions. Given a pair of operators H_1 and H_2 a kernel function is a function $\Psi(v,w)$ satisfying

$$H_1(v)\Psi(v,w) = H_2(w)\Psi(v,w),$$

where v and w may vary over spaces of different dimensions.
For the case at hand, we use kernel functions $\Psi_N^\sharp(x,y)$ that connect Hamiltonians $H_{k,\delta}(x_1,\dots,x_N)$ in N variables to a sum of two Hamiltonians in $N-1$ variables $y_1,\dots,y_N-1$. More specifically, they satisfy the key identities

$$H_{k,\delta}^{(N)}(x)\Psi_N^\sharp(x,y) = \left(H_{k,\delta}^{(N-1)}(-y) + H_{k-1,\delta}^{(N-1)}(-y)\right)\Psi_N^\sharp(x,y), \tag{2}$$

where $k = 1,\dots,N$, $\delta = +,-$, and $H_{N,\delta}^{(N-1)} \equiv 0$, $H_{0,\delta}^{(N-1)} \equiv 1$.
Moreover, $\Psi_N^\sharp$ has an explicit expression in terms of the so-called hyperbolic gamma function $G(z) \equiv G(a_+,a_-;z)$:

$$\Psi(x,y) = \mathcal{S}_N^\sharp(x,y)[W(x)W(y)]^{1/2}$$

with

$$\mathcal{S}_N^\sharp(b;x,y) \equiv \prod_{j=1}^N \prod_{k=1}^{N-1} \frac{G(x_j - y_k - ib/2)}{G(x_j - y_k + ib/2)}$$

and

$$W(x) = 1/C(x)C(-x), \quad C(b;x) = \prod_{1\le j<k\le N} \frac{G(x_j - x_k + ia - ib)}{G(x_j - x_k + ia)}.$$

The name 'hyperbolic gamma function' is motivated by the fact that $G(z)$ is a (minimal) solution of the analytic difference equations

$$\frac{G(z + ia_\delta/2)}{G(z - ia_\delta/2)} = 2\cosh(\pi z/a_{-\delta}), \quad \delta = +,-.$$

A recursive scheme

Key observation: The connection (2) between the $N-1$ and N variable cases allows us to set up a recursive scheme to explicitly construct N-variable joint eigenfunctions of the $2N$ Hamiltonians $H_{k,\delta}$, adding one more variable in each step.
Indeed, assume that we have a function $\mathcal{F}_{N-1}(x,y)$ that satisfies the eigenvalue equations

$$H_{k,\delta}^{(N-1)}(x)\mathcal{F}_{N-1}(x,y) = S_k(e^{2\pi y_1/a_\delta},\dots,e^{2\pi y_{N-1}/a_\delta})\mathcal{F}_{N-1}(x,y), \tag{3}$$

where $S_k^{(M)}(a_1,\dots,a_M)$ denotes the elementary symmetric function of $M-1$ variables $a_1,\dots,a_M$. Consider the function $\mathcal{F}_N$ given (formally) by

$$\mathcal{F}_N(x,y) = \frac{\exp\left(\frac{2\pi i}{a_+a_-} y_N \sum_{j=1}^N x_j\right)}{(N-1)!} \times \int_{\mathbb{R}^{N-1}} \Psi_N^\sharp(x,z)\mathcal{F}_{N-1}(z,(y_1-y_N,\dots,y_{N-1}-y_N))dz.$$

Then, using essentially only (2) and a standard recurrence relation for the symmetric functions $S_k^{(M)}$ one can verify (formally) that $\mathcal{F}_N$ satisfies the eigenvalue equation (3) for $N-1 \to N$. The details on how to make this precise can be found in reference [3].

References

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