Lassalle–Nekrasov correspondence between rational and trigonometric Calogero-Moser systems

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Calogero-Moser systems

Integrable N-particle systems in one dimension w/ (defining) Hamiltonian of the form

$$H = \sum_{j=1}^{N} p_j^2 + U(x_1, \ldots, x_N) \quad (p_j = -i\partial_{x_j}).$$

- Rational w/ harmonic confinement (Calogero, 1971):

$$U_R(x_1, \ldots, x_N) = \sum_{1 \le i < j \le N} \frac{\gamma}{(x_i - x_j)^2} + \omega^2 x^2, \quad x^2 = \sum_{i=1}^{N} x_i^2.$$

- Trigonometric (Sutherland, 1971):

$$U_T(x_1, \ldots, x_N) = \sum_{1 \le i < j \le N} \frac{\gamma a^2}{\sin^2 a(x_i - x_j)}.$$

- Moser (1975) proved integrability at the classical level by obtaining Lax representations.
- Olshanetsky & Perelomov (1977) established quantum integrability and introduced root system generalisations.
  (The above systems correspond to A_{N-1}.)

Lassalle–Nekrasov Correspondence

The rational ($U = U_R$) and trigonometric ($U = U_T$) dynamics are very different:

- Rational: the system is isochronous, i.e. all solutions are periodic w/ the same period $2\pi/\omega$.
- Trigonometric: the motion is much more complicated.

In a surprising development, Nekrasov (1997) showed that the two systems are essentially equivalent!

- More precisely, he constructed a symplectomorphism

$$\pi : M_R \to M_T,$$

  where $M_{R/T}$ denotes the phase space of the rational/trigonometric system, mapping integrals to integrals.
- In particular, the rational Hamiltonian is mapped to the trigonometric momentum!

Lassalle–Nekrasov Correspondence

Explains an earlier construction of Lassalle (1991).

- Specifically, he constructed multivariable Hermite polynomials from Jack polynomials.
- Form an orthogonal basis in the (complex) algebra of symmetric polynomials.
- Can be interpreted as a correspondence between eigenfunctions of the rational w/ harmonic confinement and trigonometric Calogero-Moser systems.

We call this equivalence the Lassalle–Nekrasov correspondence.

Aim: describe a generalisation of the (quantum) correspondence from the symmetric to the much wider quasi-invariant setting.

Reminder: Classical Hermite polynomials

'Probabilistic' convention: monic polynomials

$$H_n(x) = x^n + \sum_{i=0}^{n-1} a_i x^{n-i}, \quad a_i \in \mathbb{Z},$$

orthogonal wrt. the Gaussian weight $w(x) = e^{-x^2/2}$.

- The (renormalised) Hermite functions $\psi_n(x) = e^{-x^2/2} H_n(x)$ satisfy

$$\left(-\frac{d^2}{dx^2} + \frac{x^2}{4}\right)\psi_n = (n + 1/2)\psi_n.$$

- Generating function:

$$e^{kx - k^2/2} = \sum_{n=0}^{\infty} H_n(x) \frac{k^n}{n!}.$$

- Alternatively, w/ the bilinear form

$$\langle p, q \rangle = (p(d/dx)q)(0), \; p, q \in \mathbb{C}[x] \;\Leftrightarrow\; \langle x^n, x^{n'} \rangle = n! \delta_{nn'},$$

  we have

$$H_n(x) = \left\langle k^n, e^{kx - k^2/2} \right\rangle.$$

Quasi-invariants

Subalgebra

$$\mathcal{Q}_m \subset \mathbb{C}[x_1, \ldots, x_N], \quad m \in \mathbb{Z}_{\geq 0},$$

consisting of polynomials $p(x_1, \ldots, x_N)$ that are permutation-invariant up to order $2m$.

- More precisely,

$$p(x) - p(s_{ij}x) \equiv 0 \mod (x_i - x_j)^{2m+1},$$

  (where $1 \leq i < j \leq N$ and s_{ij} denotes the transposition $i \leftrightarrow j$),
- or equivalently,

$$(\partial/\partial x_i - \partial/\partial x_j)^{2k-1} p(x) \equiv 0, \quad x_i = x_j, \quad k = 1, \ldots, m,$$

  (where $1 \leq i < j \leq N$).
- Interpolate between $\mathbb{C}[x_1, \ldots, x_N]^{S_N}$ and $\mathbb{C}[x_1, \ldots, x_N]$:

$$\mathcal{Q}_\infty \equiv \mathbb{C}[x_1, \ldots, x_N]^{S_N} \subset \mathcal{Q}_m \subset \mathbb{C}[x_1, \ldots, x_N] = Q_0.$$

- In the simplest nontrivial case $N = 2$,

$$\mathcal{Q}_m = \mathbb{C}\langle x_1 + x_2, (x_1 - x_2)^2, (x_1 - x_2)^{2m+1}\rangle.$$

Multidimensional Baker-Akhiezer function

Introduced by Chalykh & Veselov (1990) to address the problem:

Describe all supercomplete commutative rings of differential operators in $\mathbb{R}^N$*, containing some Schrödinger operator* $H = -\sum_{i=1}^N \partial_{x_i}^2 + U(x)$.

(In other words, H should have at least $N+1$ commuting (algebraically) independent integrals.)

Specifically, the BA function $\phi(x,k)$, $x, k \in \mathbb{C}^N$, is uniquely determined by:

- $\phi(x,k)$ is of the form

$$\phi(x,k) = P(x,k)e^{(x,k)}$$

  for some polynomial (in x)

$$P(x,k) = A_m(x)A_m(k) + \text{lower degree terms},$$

  where

$$A_m(x) = \prod_{1 \le i < j \le N} (x_i - x_j)^m.$$

- $\phi(x,k)$ is m-quasi-invariant (in x).

Baker-Akhiezer function

Properties (Chalykh & Veselov, 1990; Chalykh, Feigin & Veselov, 1999):

- $\phi(x,k) = \phi(k,x)$,
- for each $q \in \mathcal{Q}_m$ exists diff. op. L_q s.t.

$$L_q \phi(x,k) = q(k)\phi(x,k),$$

- in particular,

$$L_{x^2} = \sum_{i=1}^{N} \partial_{x_i}^2 - \sum_{1\le i<j\le N} \frac{m}{x_i - x_j}(\partial_{x_i} - \partial_{x_j})$$

- $[L_q, L_{q'}] = 0$ for all $q, q' \in \mathcal{Q}_m$,
- $L_q \mathcal{Q}_m \subset \mathcal{Q}_m$ for each $q \in \mathcal{Q}_m$.

Rmk:

$$L_{x^2} = -A_m(x)^{-1} \circ H \circ A_m, \quad H = -\sum_{i=1}^{N} \partial_{x_i}^2 + \sum_{1\le i<j\le N} \frac{2m(m+1)}{(x_i - x_j)^2}.$$

m-Hermite polynomials

Recall the rational Calogero-Moser Hamiltonian w/ harmonic confinement:

$$H_R = -\sum_{i=1}^{N} \partial_{x_i}^2 + \sum_{1 \le i < j \le N} \frac{2m(m+1)}{(x_i - x_j)^2} + \frac{1}{4} \sum_{i=1}^{N} x_i^2,$$

taking $m \in \mathbb{Z}_{\ge 0}$.
(Here $\gamma = 2m(m+1)$ and $\omega = 1/2$).

Rmk: For $N = 1$, we recover the harmonic oscillator

$$H_R = -\frac{d^2}{dx^2} + \frac{x^2}{4}.$$

Convenient to work w/

$$\begin{aligned} L_R &:= -\Psi_0(x)^{-1} \big(H_R + mN(N-1)/2 - N/2 \big) \Psi_0(x) \\ &= \sum_{i=1}^{N} \partial_{x_i}^2 - \sum_{1 \le i < j \le N} \frac{m}{x_i - x_j} (\partial_{x_i} - \partial_{x_j}) - \sum_{i=1}^{N} x_i \partial_{x_i}, \end{aligned}$$

where $\Psi_0(x) = A_m(x)^{-1} \exp(-x^2/4)$.

m-Hermite polynomials

Consider the bilinear form

$$\langle p, q \rangle_m := \phi(0,0)^{-1}(L_p q)(0), \quad p, q \in \mathcal{Q}_m.$$

Rmk: For $N = 1$,

$$\mathcal{Q}_m = \mathbb{C}[x], \quad \phi(0,0) = e^{xk}|_{x=k=0} = 1, \quad \langle p, q \rangle = \big(p(d/dx)q\big)(0).$$

Definition

We let

$$F(x,k) = \phi(x,k)\exp(-k^2/2)$$

and define a 'Hermitisation' map $\chi_H : \mathcal{Q}_m \to \mathcal{Q}_m$, $q \mapsto H_q$ by

$$H_q(x) = \langle q, F(x, \cdot) \rangle_m.$$

If q is homogeneous, we call H_q a m-Hermite polynomial.

Proposition

- *If $q \in \mathcal{Q}_m$ is homogeneous, then*

$$H_q = q + \textit{lower degree terms.}$$

- *For any homogenous basis q_i in $\mathcal{Q}_m$, the m-Hermite polynomials H_{q_i} form a basis in $\mathcal{Q}_m$.*

Proof.

Combining

$$L_q(k)\phi(x,k) = q(x)\phi(x,k), \quad H_q(x) = \left(L_q(k)\phi(x,k)\,\mathbf{exp}(-k^2/2)\right)\Big|_{k=0},$$

we obtain the first claim.

Quasi-invariance of $H_q(x)$ follows from that of $\phi(x,k)$. Proceeding by induction in the degree d, we thus arrive at the second claim. □

m-Hermite polynomials

Proposition

For homogenous $q \in \mathcal{Q}_m$, we have

$$L_R H_q = -(\deg q) H_q .$$

Proof.

From the definition of $\phi(x,k)$, it is readily seen that $\phi(tx,k) = \phi(x,tk)$. Taking the limit $t \to 0$ in $(\phi(tx,k) - \phi(x,tk))/t = 0$, we obtain

$$E_x \phi(x,k) - E_k \phi(x,k) = 0, \quad E_z = \sum_{i=1}^{N} z_i \partial_{z_i} .$$

Combining this identity w/ $L_x \phi(x,k) = k^2 \phi(x,k)$, we deduce

$$L_{R,x} F(x,k) = (L_x - E_x)\phi(x,k) \exp(-k^2/2) = -E_k F(x,k).$$

Note that E is self-adj., since homogeneous components of $\mathcal{Q}_m$ of different degrees are orthogonal. Hence,

$$(L_R \chi_H)(q) = \langle q(\cdot), L_R F(x,\cdot)\rangle_m = -\langle Eq(\cdot), F(x,\cdot)\rangle_m = -(\chi_H E)(q).$$

m-Hermite polynomials

Further properties:

- Introducing the bilinear form

$$\{p,q\}_m = \frac{1}{(2\pi)^{N/2}} \int_{i\xi+\mathbb{R}^N} \frac{p(x)q(x)}{\prod_{1\le i<j\le N}(x_i-x_j)^{2m}} e^{-x^2/2} dx, \quad p,q \in \mathcal{Q}_m,$$

  we have

$$\{H_p, H_q\}_m = \langle p, q\rangle_m,$$

  (independent of $\xi \in \mathbb{R}^N$ as long as $\xi_i \neq \xi_j$ for all $1 \le i < j \le N$).

- For $q \in \mathcal{Q}_m$,

$$H_q(x) = \frac{\exp(x^2/2)}{(2\pi)^{N/2}} \int_{i\xi+\mathbb{R}^N} \frac{q(-iz)\phi(iz,x)}{\prod_{1\le i<j\le N}(z_i-z_j)^{2m}} e^{-z^2/2} dz$$

- and

$$H_q = \exp(-L/2)q, \quad L = \sum_{i=1}^N \partial_{x_i}^2 - \sum_{1\le i<j\le N} \frac{m}{x_i-x_j}(\partial_{x_i}-\partial_{x_j}).$$

Lassalle–Nekrasov correspondence

Recall: The Hermitisation map

$$\chi_H : \mathcal{Q}_m \to \mathcal{Q}_m, \;\; q \mapsto H_q(x) := \langle q, F(x, \cdot)\rangle_m$$

intertwines between

$$\begin{aligned} L_R &:= -\Psi_0(x)^{-1}\big(H_R + mN(N-1)/2 - N/2\big)\Psi_0(x) \\ &= \sum_{i=1}^N \partial_{x_i}^2 - \sum_{1\le i<j\le N} \frac{m}{x_i - x_j}(\partial_{x_i} - \partial_{x_j}) - \sum_{i=1}^N x_i \partial_{x_i}, \end{aligned}$$

and

$$E = \sum_{i=1}^N x_i \partial_{x_i}$$

Rmk: Writing $x_j = e^{2iz_j}$, we get

$$E = \frac{1}{2}\sum_{j=1}^N (-i\partial_{z_j}),$$

which can be viewed as an integral (total momentum) of the trigonometric Calogero-Moser system!

Lassalle–Nekrasov correspondence

More generally, consider Heckman's (1991) 'global' Dunkl operators

$$\mathcal{D}_i = x_i D_i - \frac{m}{2} \sum_{j \neq i} (s_{ij} - 1),$$

w/ the original Dunkl (1981) operators

$$D_i = \partial_{x_i} + m \sum_{j \neq i} \frac{1}{x_i - x_j} (s_{ij} - 1).$$

Properties (Heckman, 1991):

- The operators $L_{T,d} := \mathrm{Res}(\mathcal{D}_1^d + \cdots + \mathcal{D}_N^d)$ commute, (where Res means restriction to $\mathbb{C}[x_1, \ldots, x_N]^{S_N}$).
- $L_{T,1} = E$ and

$$L_{T,2} = -\frac{1}{4} \Phi_0(z)^{-1} \left(H_T - m^2 N(N^2 - 1)/3 \right) \Phi_0(z),$$

w/ $x_j = e^{i2z_j}$ and

$$\Phi_0(z) = \prod_{1 \leq i < j \leq N} \sin^{-m}(z_i - z_j), \quad H_T = -\sum_{i=1}^{N} \partial_{z_i}^2 + \sum_{1 \leq i < j \leq N} \frac{2m(m+1)}{\sin^2(z_i - z_j)}.$$

Lassalle–Nekrasov correspondence

Theorem

Following Baker & Forrester (1997), Let

$$L_{R,d} = L_{T,d} + \sum_{l=1}^{d} \frac{(-1)^l}{2^l \cdot l!} \mathrm{ad}_L^l(L_{T,d}), \quad d = 1, 2, \ldots,$$

w/ $\mathrm{ad}_L(L_{T,d}) = [L, L_{T,d}]$.
Then the diagram

$$\begin{array}{ccc} \mathcal{Q}_m & \xrightarrow{\chi_H} & \mathcal{Q}_m \\ {\scriptstyle L_{T,d}}\downarrow & & \downarrow{\scriptstyle L_{R,d}} \\ \mathcal{Q}_m & \xrightarrow{\chi_H} & \mathcal{Q}_m \end{array}$$

is commutative for all $d = 1, 2, \ldots$

Rmk: A direct computation reveals $L_{R,1} = -L_R$.

Since $\mathrm{im}(\chi_H) = \mathcal{Q}_m$, we have the following:

Corollary

The operators $L_{R,d}$ commute and are thus quantum integrals of the rational Calogero-Moser system w/ harmonic confinement.

Concluding remarks

- The proof relies on a remarkable symmetry property of the rational Baker-Akhiezer function:

$$L_{T,d}(x)\phi(x,k) = L_{T,d}(k)\phi(x,k), \quad d = 1, 2, \ldots$$

- Recall: The rational Cherednik algebra H_m can be identified w/ the algebra generated by x_i, D_i and s_{ij}.
  The map $L_{T,d} \mapsto L_{R,d}$ is essentially given by the following automorphism of H_m:

$$x_i \mapsto x_i - D_i, \quad D_i \mapsto D_i, \quad s_{ij} \mapsto s_{ij}.$$

  (introduced by Etingof & Ginzburg, 2002).
- The above results are naturally associated w/ the positive roots

$$A_{N-1+} = \{e_i - e_j : 1 \leq i < j \leq N\} \subset \mathbb{R}^N,$$

  taken w/ multiplicity $m \in \mathbb{Z}_{\geq 0}$. Part of our results generalise to all configurations of vectors in $\mathbb{R}^N$ w/ multiplicities admitting the rational Baker-Akhiezer function, (which includes all Coxeter configurations).

Reference

M. V. Feigin, M. A. H. & A. P. Veselov.
Quasi–invariant Hermite polynomials and Lassalle–Nekrasov correspondence.
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