Product formulas for conical functions

(Joint work w/ S. Ruijsenaars)

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Gustav Ferdinand Mehler (1835–1895)

Mathematics teacher at Elbinger Gymnasium, now in northern Poland. Wrote both textbooks (e.g. Hauptsätze der Elementar-Mathematik, $>$ 19 eds.) and research papers.

Influenced by Dirichlet, Mehler worked mainly on potential theory.

Ueber die Vertheilung der statischen Elektricität in einem von zwei Kugelkalotten begrenzten Körper.

(Von Herrn *F. G. Mehler* zu Danzig.)

Das Problem der Elektricitätsvertheilung in einem von zwei sich nicht schneidenden Kugelflächen begrenzten Körper ist bekanntlich schon von *Poisson* für den besonderen Fall gelöst worden, dass der Körper aus zwei vollen Kugeln besteht und keine äusseren Kräfte auf ihn einwirken; seine allgemeine und vollständige Lösung aber hat es durch eine im Jahre 1862 erschienene Schrift des Herrn *C. Neumann* erhalten, deren Methoden und Hauptresultate man auch in diesem Journal (Bd. 62, p. 36 — 49) angegeben findet. Meine Absicht ist es, den bisher, so viel ich weiss, noch unerledigten Fall in Betracht zu ziehen, wo die Grenzen des Körpers durch zwei sich schneidende Kugelflächen bestimmt sind. Jeden der vier gesonderten Räume, in welche zwei solche Flächen den ganzen unendlichen Raum theilen, wird man nach Belieben als die Elektricität leitend oder nicht leitend voraussetzen können. Welche Anordnung man aber in dieser Beziehung auch treffen, und wo auch immer in dem nichtleitenden Raume der Sitz der vertheilend wirkenden elektrischen Massen sein möge, so sind doch alle daraus entspringenden Probleme, nach dem jetzigen Stande der allgemeinen Theorie, als vollständig gelöst zu betrachten *), sobald man kennt:

I. Die Dichtigkeit der elektrischen Schicht, welche durch einen elektrischen Massenpunkt auf einem von zwei Kugelkalotten eingeschlossenen Körper hervorgerufen wird und gebunden bleibt, wenn dieser Körper mit einem unendlich grossen Leiter in leitende Verbindung gebracht wird.

II. Die durch Influenz eines elektrischen Massenpunktes erzeugte Vertheilung in einem Leiter, der von innen durch zwei Kugelkalotten begrenzt, nach aussen hin unbegrenzt ist.

III. Die Vertheilung einer gegebenen Elektricitätsmenge auf einem isolirten Leiter der ersten Art.

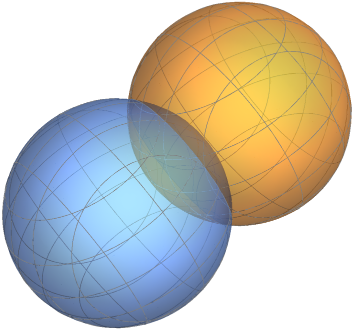
Bei der Behandlung dieser Aufgaben, von denen übrigens die dritte,

*) Man vergleiche die beiden Aufsätze des Herrn *Lipschitz* in Bd. 58, p. 1 — 53 und 152 — 173 dieses Journals.

Mehler's Dirichlet problem

The conical function was introduced by Mehler in 1868 when solving the Dirichlet problem for a domain bounded by two intersecting spheres.

Mehler was motivated by the electrostatic interpretation of the problem.



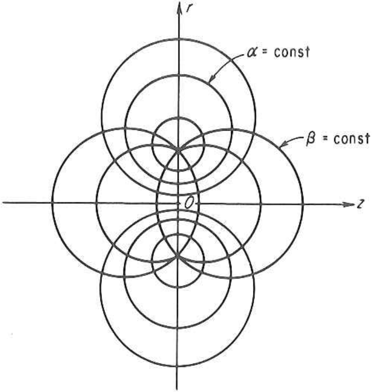
Toroidal coordinates

Obtained by rotating a bipolar coordinate system about the axis separating its foci ($x = \pm 1$, $z = 0$):

$$x = \frac{\sinh\alpha\cos\varphi}{\cosh\alpha - \cos\beta},$$
$$y = \frac{\sinh\alpha\sin\varphi}{\cosh\alpha - \cos\beta},$$
$$z = \frac{\sin\beta}{\cosh\alpha - \cos\beta},$$

where

$$0 \leqslant \alpha < \infty, \quad -\pi < \beta, \varphi \leqslant \pi.$$



Laplace's equation in toroidal coordinates

Suppose $\Delta U = 0$. Substituting

$$U = \sqrt{2\cosh\alpha - 2\cos\beta}\, V,$$

we find that

$$\frac{1}{\sinh\alpha}\frac{d}{d\alpha}\left(\sinh\alpha\frac{dV}{d\alpha}\right) + \frac{1}{4}V + \frac{\partial^2 V}{\partial\beta^2} + \frac{1}{\sinh^2\alpha}\frac{\partial^2 V}{\partial\varphi^2} = 0. \qquad (1)$$

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Proposition

A function

$$V(\alpha,\beta,\varphi) = K(\alpha)L(\beta)\Phi(\varphi)$$

satisfies (1) if

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0, \qquad \frac{d^2 L}{d\beta^2} - k^2 L = 0,$$

$$\frac{1}{\sinh\alpha}\frac{d}{d\alpha}\left(\sinh\alpha\frac{dK}{d\alpha}\right) + \left(k^2 + \frac{1}{4} - \frac{m^2}{\sinh^2\alpha}\right)K = 0.$$

The conical function

The conical function K_μ^m, introduced by Mehler in 1868, is given by

$$\begin{aligned} K_k^m(\alpha) &= P_{-1/2+ik}^{-m}(\cosh\alpha) \\ &= \left(\frac{4}{\pi}\right)^{1/2} \frac{\Gamma(m+1/2)(2\sinh\alpha)^m}{\prod_{\delta=+,-}\Gamma(i\delta k+m+1/2)} \\ &\quad \times \int_{-\infty}^{\infty} \frac{e^{ikt}}{\prod_{\delta=\pm}\left[2\cosh(t+\delta\alpha)/2)\right]^{1/2+m}}\,dt, \end{aligned}$$

(where P_τ^{-m} is a Legendre function).

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Proposition

$K_k^m(\alpha)$ satisfies the ODE

$$\frac{1}{\sinh\alpha}\frac{d}{d\alpha}\left(\sinh\alpha\frac{dK}{d\alpha}\right) + \left(k^2+\frac{1}{4}-\frac{m^2}{\sinh^2\alpha}\right)K = 0.$$

The (generalized) Mehler-fock transform

Let

$$w(g;r) = (2\sinh r)^{2g}, \quad \widehat{w}(g;k) = 4\Gamma(g)^2 \Big/ \prod_{\delta=+,-} \Gamma(i\delta k)\Gamma(i\delta k + g),$$

and consider

$$F(g;r,k) = \left(\frac{\pi}{4}\right)^{1/2} \frac{\prod_{\delta=+,-} \Gamma(i\delta k + g)}{\Gamma(g)(2\pi \sinh r)^{g-1/2}} K_k^{g-1/2}(r).$$

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Theorem (Mehler 1868 and Fock 1943)

For $\psi \in C_0^\infty((0,\infty))$, we have

$$\psi(r) = \left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \widehat{\psi}(k) F(g;r,k) \widehat{w}(g;k) dk,$$

with

$$\widehat{\psi}(k) = \left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \psi(r) F(g;r,k) w(g;r) dr.$$

Dual convolution product

Introducing

$$\mathcal{K}(k,p,q) \equiv \big(F(\cdot,k)F(\cdot,p)\big)\widehat{\ }(q)$$
$$= \left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty F(s,k)F(s,p)F(s,q)w(s)ds,$$

we have the product formula

$$F(r,k)F(r,p) = \left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \mathcal{K}(k,p,q)F(r,q)\widehat{w}(q)dq.$$

Dual convolution product

For $\phi, \psi \in L^1((0,\infty), dk)$, let

$$(\phi * \psi)(q) = \left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \int_0^\infty \phi(k)\psi(p)\mathcal{K}(k,p,q)\widehat{w}(k)\widehat{w}(p)dkdp,$$

and let

$$\check{\phi}(r) = \left(\frac{1}{2\pi}\right)^{1/2} \int_0^\infty \phi(k)F(g;r,k)\widehat{w}(g;k)dk.$$

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Proposition

$$(\phi * \psi)\check{}\,(r) = \check{\phi}(r)\check{\psi}(r)$$

Dual convolution product

Proof:

$$(\phi * \psi)^{\vee}(r)$$

Dual convolution product

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$$(\phi * \psi)\check{}(r)$$
$$= \frac{1}{2\pi} \int_0^\infty \left(\int_0^\infty \int_0^\infty \phi(k)\psi(p)\mathcal{K}(k,p,q)\widehat{w}(k)\widehat{w}(p)dkdp \right) F(r,q)\widehat{w}(q)dq$$

Dual convolution product

Proof:

$$
\begin{aligned}
&(\phi * \psi)\check{}(r)\\
&= \frac{1}{2\pi}\int_0^\infty \left(\int_0^\infty \int_0^\infty \phi(k)\psi(p)\mathcal{K}(k,p,q)\widehat{w}(k)\widehat{w}(p)dkdp\right) F(r,q)\widehat{w}(q)dq\\
&= \frac{1}{2\pi}\int_0^\infty \int_0^\infty \phi(k)\psi(p)\left(\int_0^\infty \mathcal{K}(k,p,q)F(r,q)\widehat{w}(q)dq\right)\widehat{w}(k)\widehat{w}(p)dkdp
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&= \left(\frac{1}{2\pi}\right)^{1/2}\int_0^\infty \phi(k)F(r,k)\widehat{w}(k)dk \cdot \left(\frac{1}{2\pi}\right)^{1/2}\int_0^\infty \psi(p)F(r,p)\widehat{w}(p)dp
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&= \left(\frac{1}{2\pi}\right)^{1/2}\int_0^\infty \phi(k)F(r,k)\widehat{w}(k)dk \cdot \left(\frac{1}{2\pi}\right)^{1/2}\int_0^\infty \psi(p)F(r,p)\widehat{w}(p)dp \\
&= \check{\phi}(r)\check{\psi}(r).
\end{aligned}
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Dual convolution product

Theorem (H. and Ruijsenaars 2015)

$$\mathcal{K}(g;k,p,q) = \frac{\widehat{w}(g;k)\widehat{w}(g;p)\widehat{w}(g;q)}{16\Gamma(g)^4} \times \prod_{\delta_1,\delta_2,\delta_3=+,-} \Gamma((g+i\delta_1 k + i\delta_2 p + i\delta_3 q)/2)$$

Rmk: For $g = 1/2$ and $g = 1$, first obtained by Mizony 1976. Positivity for $(k,p,q) \in (0,\infty)^3$ first proved by Flensted-Jensen and Koornwinder 1979.

More product formulas

Theorem

$$F(r,k)F(s,k) = \frac{\prod_{\delta=+,-}\Gamma(g+i\delta k)}{2^{2g+1}\Gamma(g)^2}$$

$$\times \int_{|r-s|}^{r+s} F(t,k)\frac{\prod_{\delta=+,-}(\cosh(r+\delta s)-\cosh t)^{g-1}}{(\sinh r \sinh s \sinh t)^{2g-1}} w(t)dt$$

Rmk: Special case of product formula for Legendre functions (see e.g. Hobson 1931 and Vilenkin 1968), and, more generally, for Jacobi functions (Koornwinder 1972). Yields a generalized translate and convolution product.

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Theorem (H. & Ruijsenaars 2015)

$$F(r,k)F(s,k) = \frac{1}{2}\int_0^{\infty} F(t,k)\frac{w(t)}{\prod_{\delta_1,\delta_2=+,-}[2\cosh((\delta_1 r+\delta_2 s+t)/2)]^g} dt$$

More product formulas

Sketch of a proof of the second formula:

- Verify that $F(r,k)$ satisfies

$$\mathcal{L}F \equiv -\frac{d^2F}{dr^2} + 2g\coth r\frac{dF}{dr} = -(k+g)^2F.$$

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  and formal self-adjointness of $\mathcal{L}(t)$ on $L^2((0,\infty), w(t)dt)$ show that the RHS also satisfies this ODE.

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- By computing the asymptotics of the RHS, deduce that $C \equiv 1$.

The hyperbolic (A_{N-1}) Calogero-Moser system

Can be defined by the Hamiltonian

$$H_N \equiv -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2g(g-1) \sum_{1 \leqslant j < k \leqslant N} \frac{1}{4\sinh^2((x_j - x_k)/2)},$$

where $N \in \mathbb{N}$ (particle number) and $g > 0$ (coupling constant).

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- Associated integrable system (N commuting PDOs):

$$H_N^1 = -i \sum_{j=1}^{N} \frac{\partial}{\partial x_j}, \quad H_N,$$

$$H_N^k = (-i)^k \sum_{j=1}^{N} \frac{\partial^k}{\partial x_j^k} + \cdots, \quad k = 3, \ldots, N.$$

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- Integrable versions for Lie algebras $B_N, \ldots, E_8$ (Olshanetsky & Perelomov, Oshima) and BC_N (Inozemtsev, Oshima) exist.
- $N > 2$ eigenfunctions: Harish-Chandra, Heckman & Opdam, Chalykh, Felder & Varchenko, Veselov,...

$N = 2$ eigenfunctions

Consider the function

$$\Psi_2(g; x, y) \equiv e^{iy_2(x_1+x_2)} \int_{-\infty}^{\infty} e^{i(y_1-y_2)z} \mathcal{K}_2^{\sharp}(g; x, z) dz,$$

with kernel

$$\mathcal{K}_2^{\sharp}(g; x, z) \equiv \frac{\left[4 \sinh^2(x_1 - x_2)\right]^{g/2}}{\prod_{j=1}^{2} \left[2 \cosh(x_j - z)\right]^g},$$

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$$g > 0, \quad x, y \in \mathbb{R}^2.$$

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Proposition

We have

$$H_2^1(x)\Psi_2(x, y) = (y_1 + y_2)\Psi_2(x, y),$$
$$H_2(x)\Psi_2(x, y) = (y_1^2 + y_2^2)\Psi_2(x, y).$$

$N = 2$ eigenfunctions

Sketch of a proof:

- By a direct computation, establish

$$\left(H_2(x) + \frac{d^2}{dz^2}\right)\mathcal{K}_2^\sharp(x,z) = 0, \quad \left(-\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \frac{d}{dz}\right)\mathcal{K}_2^\sharp(x,z) = 0.$$

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- Deduce that

$$\begin{aligned}
&H_2(x)\Psi_2(x,y) \\
&= e^{iy_2(x_1+x_2)}\left(H_2(x) - 2iy_2\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) + 2y_2^2\right)\int_{-\infty}^{\infty} e^{i(y_1-y_2)z}\mathcal{K}_2^\sharp(x,z)\,dz \\
&= e^{iy_2(x_1+x_2)}\int_{-\infty}^{\infty} e^{i(y_1-y_2)z}\left(-\frac{d^2}{dz^2} + 2iy_2\frac{d}{dz} + 2y_2^2\right)\mathcal{K}_2^\sharp(x,z)\,dz \\
&= e^{iy_2(x_1+x_2)}\int_{-\infty}^{\infty} \mathcal{K}_2^\sharp(x,z)\left(-\frac{d^2}{dz^2} - 2iy_2\frac{d}{dz} + 2y_2^2\right)e^{i(y_1-y_2)z}\,dz \\
&= (y_1^2 + y_2^2)\Psi_2(x,y).
\end{aligned}$$

$N = 2$ eigenfunctions

Taking $z \to z + (x_1 + x_2)/2$, we see that

$$\Psi_2(g; x, y) = e^{i(y_1+y_2)(x_1+x_2)/2}\Psi(g; x_1 - x_2, y_1 - y_2),$$

with

$$\Psi(g; r, k) \equiv [2\sinh r]^g \int_{-\infty}^{\infty} \frac{e^{iks}}{\prod_{\delta=+,-}[2\cosh(s + \delta r/2)]^g} ds.$$

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Proposition

$$\Psi(g; r, k) = \left(\frac{\pi}{4}\right)^{1/2} \frac{(2\sinh r)^{1/2} \prod_{\delta=+,-} \Gamma(i\delta k + g)}{\Gamma(g)} K_k^{g-1/2}(r)$$

Rmk: Follows from a direct comparison with integral representation for K_k^m.

Integral operators

Define

$$\mathcal{J}_2 : L^2(G_2) \to L^2(G_2), \quad G_2 \equiv \{x \in \mathbb{R}^2 : x_2 < x_1\},$$

by

$$(\mathcal{J}_2(g)\Psi)(x) = \int_{G_2} \Psi(z) \frac{[2\sinh(x_1 - x_2)]^{g/2}[2\sinh(z_1 - z_2)]^{g/2}}{\prod_{j,k=1}^{2}[2\cosh(x_j - z_k)]^g} dz.$$

Theorem (H. & Ruijsenaars 2015)

We have

$$(\mathcal{J}_2\Psi_2(\cdot, y))(x) = 2\mu(y)\Psi_2(x, y),$$

with

$$\mu(g; y) \equiv \frac{\prod_{j=1}^{2}\prod_{\delta=+,-}\Gamma((i\delta y_j + g)/2)}{4\Gamma(g)^2}.$$

Rmk: Obtained from the latter product formula, and the former yields a similar result. As a corollary, we get that $\mathcal{J}_2$ is bounded and self-adjoint.

$N > 2$ eigenfunctions

- Exist representations by multidimensional integrals whose integrands are elementary functions (Borodin & Gorin, Felder & Veselov, H. & Ruijsenaars and Yi).
- We believe that corresponding integral operators and equations can be generalised to $N > 2$...

Relativistic/hyperbolic generalization

The hyperbolic gamma function is given by

$$G(a_+, a_-; z) \equiv \exp(ig(a_+, a_-; z)),$$

with

$$g(z) \equiv \int_0^\infty \frac{dy}{y} \left(\frac{\sin 2yz}{2\sinh(a_+y)\sinh(a_-y)} - \frac{z}{a_+a_-y} \right),$$

for $|\mathrm{Im}\,(z)| < (a_+ + a_-)/2$.

Relativistic/hyperbolic generalization

The hyperbolic gamma function is given by

$$G(a_+, a_-; z) \equiv \exp(ig(a_+, a_-; z)),$$

with

$$g(z) \equiv \int_0^\infty \frac{dy}{y} \left(\frac{\sin 2yz}{2\sinh(a_+ y)\sinh(a_- y)} - \frac{z}{a_+ a_- y} \right),$$

for $|\mathrm{Im}\,(z)| < (a_+ + a_-)/2$.

Proposition (Ruijsenaars 1997)

$$\frac{G(z + ia_\delta/2)}{G(z - ia_\delta/2)} = 2\cosh(\pi z / a_{-\delta}), \qquad \delta = +, -$$

Relativistic/hyperbolic generalization

Consider the function

$$J(b; x, y) \equiv \int_{\mathbb{R}} \exp\left(i2\pi zy/(a_+ a_-)\right) \prod_{\delta=+,-} \frac{G(z+\delta x/2 - ib/2)}{G(z+\delta x/2 + ib/2)}\, dz,$$

and (analytic) difference operators

$$A_\delta(b; z) \equiv \frac{\sinh(\pi(z-ib)/a_\delta)}{\sinh(\pi z/a_\delta)} \exp(-ia_{-\delta}\partial_z) + \frac{\sinh(\pi(z+ib)/a_\delta)}{\sinh(\pi z/a_\delta)} \exp(ia_{-\delta}\partial_z), \quad \delta = +, -.$$

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Proposition (Ruijsenaars 2011)

For $\delta = +,-$, we have

$$A_\delta(b;x)J(b;x,y) = 2\cosh(\pi y/a_\delta)J(b;x,y),$$
$$A_\delta(2a-b;y)J(b;x,y) = 2\cosh(\pi x/a_\delta)J(b;x,y).$$

Relativistic/hyperbolic generalization

Proposition (H. & Ruijsenaars 2015)

$$\lim_{\beta \to 0} J(\pi, \beta; \beta g; r, \beta k) = F(g; r, k)$$

Rmk: The non-relativistic limit when taking $\beta = 1/mc$, with m particle mass and c speed of light.

Relativistic/hyperbolic generalization

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Rmk: The non-relativistic limit when taking $\beta = 1/mc$, with m particle mass and c speed of light.

Theorem (H. & Ruijsenaars 2015)

We have

$$J(x, v)J(y, v) = \frac{1}{2} \int_0^\infty w(z) J(z, v) \prod_{\delta_1, \delta_2, \delta_3 = +,-} G((\delta_1 x + \delta_2 y + \delta_3 z)/2)\, dz,$$

with

$$w(b; z) \equiv \prod_{\delta = +,-} \frac{G(\delta z + i(a_+ + a_-)/2)}{G(\delta z + i(a_+ + a_-)/2 - ib)}.$$

Rmk: Representation-theoretical interpretation?

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