Deformed elliptic Ruijsenaars operators and hypergeometric transformation formulas

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Virtual Integrable Systems Seminars
24 February 2021

Ruijsenaars' elliptic difference operators

For $n \in \mathbb{N}$,

$$D_n^{(k)} = \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I|=k}} \prod_{i \in I, j \notin I} \frac{[x_i - x_j + \kappa]}{[x_i - x_j]} \cdot T_x^{\delta I} \quad (k = 1, \ldots, n).$$

Notation:

- δ and κ are (complex) parameters,
- $[z] = Ce^{cz^2}\sigma(z \mid \omega_1, \omega_2)$ w/ degenerations:

$$[z] = \sin(\pi z/\omega) \quad \text{(trigonometric)},$$

$$[z] = \sinh(\pi z/\omega) \quad \text{(hyperbolic)},$$

$$[z] = z \quad \text{(rational)},$$

  (where σ denotes the Weierstrass sigma function),
- $T_x^{\delta I} = \prod_{i \in I} T_{x_i}^{\delta}$, w/

$$T_{x_i}^{\delta} f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, x_i + \delta, \ldots, x_n).$$

Ruijsenaars' elliptic difference operators

- Introduced by Ruijsenaars (1987), who proved commutativity:

$$\left[D_n^{(k)}, D_n^{(l)}\right] = 0, \quad \forall k, l = 1, \ldots, n.$$

- Calogero–Moser–Sutherland operators are limiting cases. For example, with $[z] = z$ and $\kappa \to \delta\kappa$,

$$D_n^{(1)} = n + \delta \sum_{i=1}^{n} \partial_{x_i} + \frac{\delta^2}{2}\left(\sum_{i=1}^{n} \partial_{x_i}^2 + 2\kappa \sum_{j\neq i} \frac{1}{x_i - x_j}\partial_{x_i}\right) + O(\delta^3), \quad \delta \to 0.$$

- Also called relativistic Calogero-Moser–Sutherland systems, since

$$H_n := D_n^{(1)}(x) + D_n^{(1)}(-x) \quad \text{(time transl.)}$$

$$P_n := D_n^{(1)}(x) - D_n^{(1)}(-x) \quad \text{(space transl.)}$$

$$B := -\sum_{i=1}^{n} x_j \quad \text{(Lorentz boost)}$$

  yield a representation of the Lie alg. of the Poincaré group in $1+1$ dimensions, see Ruijsenaars (1987).

- Intimate connections w/ integrable (quantum) field theories. (For example, when $[z] = \sinh(\pi z/\omega)$ joint eigenfuncs. of $D_n^{(k)}$ reproduce scattering in the quantum sine-Gordon model (for suitable δ, κ); see H. & Ruijsenaars (2020).)

Noumi and Sano's operators

Consider the commutative alg.

$$\mathcal{R}_n = \mathbb{C}\left[D_n^{(1)}, \ldots, D_n^{(n)}\right].$$

Noumi & Sano (2020) introduced the difference operators

$$H_n^{(k)} = \sum_{\substack{\mu\in\mathbb{N}^n \\ |\mu|=k}} \prod_{1\le i<j\le n} \frac{[x_i - x_j + (\mu_i - \mu_j)\delta]}{[x_i - x_j]} \cdot \prod_{i,j=1}^{n} \frac{[x_i - x_j + \kappa]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}} \cdot T_x^{\delta\mu} \quad (k \in \mathbb{N}),$$

and proved:

- $\sum_{k+l=K}(-1)^k[k\kappa + l\delta]D_n^{(k)}H_n^{(l)} = 0$, $K \in \mathbb{N}$, (Wronski relations),
- $\mathcal{R}_n = \mathbb{C}\left[H_n^{(1)}, \ldots, H_n^{(n)}\right]$.

Notation:

- $[z]_k = [z][z+\delta]\cdots[z+(k-1)\delta]$,
- $T_x^{\delta\mu} = \prod_{i=1}^{n}\left(T_{x_i}^{\delta}\right)^{\mu_i}$.

Deformed elliptic Ruijsenaars operators

Ruijsenaars' and Noumi & Sano's operators can be unified in a family of <u>commutative</u> difference operators $H^{(k)}_{m,r}(x,y;\delta,\kappa)$ *(*$k\in\mathbb{N}$*) in two sets of variables* $x=(x_1,\ldots,x_m)$ *and* $y=(y_1,\ldots,y_r)$*.*

Deformed elliptic Ruijsenaars operators

We introduce the difference operators

$$H_{m,r}^{(k)} = \sum_{\substack{\mu \in \mathbb{N}^m, I \subset \{1,\dots,r\} \\ |\mu|+|I|=k}} C_{\mu,I}(x,y)\, T_x^{\delta\mu}\, T_y^{-\kappa I} \quad (k \in \mathbb{N}),$$

with

$$C_{\mu,I}(x,y) = (-1)^{|I|} \prod_{1\le i<j\le m} \frac{[x_i - x_j + (\mu_i - \mu_j)\delta]}{[x_i - x_j]} \cdot \prod_{i,j=1}^{m} \frac{[x_i - x_j + \kappa]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}}$$

$$\cdot \prod_{i\in I, j\notin I} \frac{[y_i - y_j - \delta]}{[y_i - y_j]}$$

$$\cdot \prod_{i=1}^{m} \left(\prod_{j\in I} \frac{[x_i - y_j - \kappa]}{[x_i - y_j + \mu_i \delta]} \prod_{j\notin I} \frac{[x_i - y_j - \delta]}{[x_i - y_j + (\mu_i - 1)\delta]} \right).$$

Obs: We have

- $H_{m,0}^{(k)}(x;\delta,\kappa) = H_m^{(k)}(x;\delta,\kappa)$,
- $H_{0,r}^{(k)}(y;\delta,\kappa) = (-1)^k D_r^{(k)}(y;-\kappa,-\delta)$.

Main results

Theorem (H., Langmann, Noumi & Rosengren)

- *We have*
$$\left[H^{(k)}_{m,r}, H^{(l)}_{m,r}\right] = 0, \quad \forall k, l \in \mathbb{N},$$
- *The operators* $H^{(1)}_{m,r},\ldots,H^{(m+r)}_{m,r}$ *are algebraically independent (for generic* δ,κ*).*

Consider the difference operators

$$D^{(k)}_{m,r}(x,y;\delta,\kappa) = H^{(k)}_{r,m}(y,x;-\kappa,-\delta) \quad (k \in \mathbb{N}),$$

(with $D^{(k)}_{m,0}(x;\delta,\kappa) = (-1)^k D^{(k)}_m(x;\delta,\kappa)$ and $D^{(k)}_{0,r}(y;\delta,\kappa) = H^{(k)}_r(y;-\kappa,-\delta)$.)

Theorem (H., Langmann, Noumi & Rosengren)

The operators $D^{(k)}_{m,r}$ *and* $H^{(k)}_{m,r}$ *are related by*

$$\sum_{k+l=K} [k\kappa + l\delta] D^{(k)}_{m,r} H^{(l)}_{m,r} = 0 \quad (K \in \mathbb{N}).$$

Main results

Fix a meromorphic solution G_δ to

$$G_\delta(z+\delta) = [z] G_\delta(z).$$

(Generically, G_δ can be constructed from Ruijsenaars' elliptic gamma function.)

Theorem (H., Langmann, Noumi & Rosengren)

Assuming that

$$(m-n)\kappa = (r-s)\delta,$$

the function

$$\begin{aligned}\Phi^{(m,r,n,s)}&(x_1,\dots,x_m;y_1,\dots,y_r;X_1,\dots,X_n;Y_1,\dots,Y_s)\\ &= \prod_{\substack{1\le i\le m\\1\le j\le n}} \frac{G_\delta(x_i+X_j-\kappa)}{G_\delta(x_i+X_j)} \cdot \prod_{\substack{1\le i\le r\\1\le j\le s}} \frac{G_{-\kappa}(y_i+Y_j+\delta)}{G_{-\kappa}(y_i+Y_j)}\\ &\qquad \cdot \prod_{\substack{1\le i\le m\\1\le j\le s}} [x_i+Y_j] \cdot \prod_{\substack{1\le i\le r\\1\le j\le n}} [y_i+X_j]\end{aligned}$$

satisfies the kernel function identities

$$H_{m,r}^{(k)}(x;y)\Phi^{(m,r,n,s)}(x;y;X;Y) = H_{n,s}^{(k)}(X;Y)\Phi^{(m,r,n,s)}(x;y;X;Y) \quad (k\in\mathbb{N}).$$

Historical interlude

Consider

$$-\sum_{i=1}^{n}\frac{1}{2}\frac{\partial^2}{\partial x_j^2}-\frac{m}{2}\sum_{i=1}^{r}\frac{\partial^2}{\partial y_i^2}+m(m+1)\sum_{1\le i<j\le n}V(x_i-x_j)$$
$$+(m+1)\sum_{i=1}^{n}\sum_{j=1}^{r}V(x_i-y_j)+(1+1/m)\sum_{1\le i<j\le r}V(y_i-y_j),$$

w/ potential function

$$V(z)=\begin{cases}1/z^2 & \text{(rational)}\\ 1/\sin^2 z & \text{(trigonometric)}\\ \wp(z) & \text{(elliptic)}\end{cases}$$

(When $m=1$ and/or $r=0$ we have an ordinary Calogero–Moser–Sutherland operator.)

- Chalykh, Feigin & Veselov (1998) proved integrability when $r=1$ and V is rational/trigonometric.
- For $n,r\in\mathbb{N}$ arbitrary and V trigonometric, the operator was introduced and studied by Sergeev (2001). Integrability proved by Sergeev & Veselov (2004).

Historical interlude

- Khodarinova (2005) established integrability for $r = 1$ and V elliptic.
- There are intimate connections with
  - Lie superalgebras (Sergeev, Seergev & Veselov),
  - Cherednik algebras (Feigin),
  - β-ensembles of random matrices (Desrosiers & Liu),
  - CFT and the fractional quantum Hall effect (Atai & Langmann),
  - ...
- Chalykh (2000, 2002) introduced analogous deformations of rational/trigonometric Ruijsenaars operators in $n+1$ variables.
- The trigonometric limit of $H^{(1)}_{m,r}$ due to Sergeev & Veselov (2009).
- Feigin and Silantyev (2014) obtained the trigonometric limit of $H^{(k)}_{m,r}$ for all $k \in \mathbb{N}$ and proved commutativity.
- The elliptic operator $H^{(1)}_{m,r}$ was first considered by Atai, H. & Langmann (2014), who established a corresponding kernel function identity.

Proof of commutativity

There are two main steps in our proof.

Step 1: We reduce $\left[H^{(k)}_{m,r}, H^{(l)}_{m,r}\right] = 0$ to the identities

$$S_k = S_{|\lambda|+r-k}, \quad \lambda \in \mathbb{N}^m, \ \ 0 \le k \le |\lambda|,$$

for

$$S_k = \sum_{\substack{0\le\mu_j\le\lambda_j, 1\le j\le m \\ P\subset\{1,\dots,r\}, |\mu|+|P|=k}} \prod_{i\in P, j\notin P} \frac{[y_i - y_j - \delta][y_i - y_j + \delta - \kappa]}{[y_i - y_j][y_i - y_j - \kappa]}$$
$$\cdot \prod_{i,j=1}^{m} \left(\frac{[x_i - x_j + \delta]_{\mu_i - \mu_j}}{[x_i - x_j + \kappa]_{\mu_i - \mu_j}} \frac{[x_i - x_j + \kappa]_{\mu_i} [x_i - x_j - \lambda_j \delta]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i} [x_i - x_j - (\lambda_j - 1)\delta - \kappa]_{\mu_i}} \right)$$
$$\cdot \prod_{i=1}^{m} \left(\prod_{j\in P} \frac{[x_i - y_j + \lambda_i \delta][x_i - y_j + (\mu_i - 1)\delta + \kappa]}{[x_i - y_j + \mu_i \delta][x_i - y_j + (\lambda_i - 1)\delta + \kappa]} \right.$$
$$\left. \cdot \prod_{j\notin P} \frac{[x_i - y_j - \delta][x_i - y_j + \mu_i \delta - \kappa]}{[x_i - y_j - \kappa][x_i - y_j + (\lambda_i - 1)\delta]} \right).$$

Proof of commutativity

Remark: We note that

$$\prod_{i,j=1}^{m} \frac{[x_i - x_j + \delta]_{\mu_i - \mu_j}}{[x_i - x_j + \kappa]_{\mu_i - \mu_j}} = \prod_{1 \le i < j \le m} \left(\frac{[x_i - x_j + (\mu_i - \mu_j)\delta]}{[x_i - x_j]} \frac{[x_i - x_j + \delta - \kappa]_{\mu_i - \mu_j}}{[x_i - x_j + \kappa]_{\mu_i - \mu_j}} \right),$$

where factors of the form $[x_i - x_j + (\mu_i - \mu_j)\delta]$ are typical of elliptic hypergeometric series related to root systems of type A. In fact, $S_k = S_{|\lambda|+r-k}$ is essentially equivalent to an elliptic hypergeometric transformation formula due to Langer, Schlosser and Warnaar (2009).

Step 2: We obtain the identity $S_k = S_{|\lambda|+r-k}$ by *multiple principal specialization* in

$$\sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=k}} \prod_{i \in I, j \notin I} \frac{[z_i - z_j - a][z_i - z_j - b]}{[z_i - z_j][z_i - z_j - a - b]} = \sum_{\substack{I \subset \{1,\dots,n\} \\ |I|=n-k}} \prod_{i \in I, j \notin I} \frac{[z_i - z_j - a][z_i - z_j - b]}{[z_i - z_j][z_i - z_j - a - b]}.$$

The latter identity is due to Ruijsenaars (1987).
(He used it to prove commutativity for his elliptic difference operators.)

Proof of commutativity

Specifically, we take

$$n = |\lambda| + r, \quad a = \delta, \quad b = \kappa - \delta,$$

and set

$$(z_1, \dots, z_n) = (x_1, x_1 + \delta, \dots, x_1 + (\lambda_1 - 1)\delta, \dots, \\ x_m, x_m + \delta, \dots, x_m + (\lambda_m - 1)\delta, y_1, \dots, y_r).$$

Further source identities

- We infer the Wronski type relation $\sum_{k+l=K}(-1)^k[k\kappa+l\delta]D_n^{(k)}H_n^{(l)}=0$ from

$$\sum_{\substack{I\subset\{1,\ldots,n\}\\|I|=k}}(-1)^{|I|}\frac{[|z|-|w|+|I|a]}{[|z|-|w|]}\prod_{i\in I,j\notin I}\frac{[z_i-z_j+a]}{[z_i-z_j]}\cdot\prod_{\substack{i\in I\\j\in\{1,\ldots,n\}}}\frac{[z_i-w_j]}{[z_i-w_j+a]}=0,$$

  where

$$|z|=\sum_{j=1}^n z_j.$$

  (This is the same identity proved and used by Noumi & Sano (2020) in the undeformed $r=0$ case.)

- We obtain the kernel identities from the Kajihara–Noumi (2003) identity

$$\sum_{\substack{I\subset\{1,\ldots,n\}\\|I|=k}}\prod_{i\in I,j\notin I}\frac{[z_i-z_j-a]}{[z_i-z_j]}\prod_{\substack{i\in I\\j\in\{1,\ldots,n\}}}\frac{[z_i+w_j+a]}{[z_i+w_j]}$$

$$=\sum_{\substack{I\subset\{1,\ldots,n\}\\|I|=k}}\prod_{i\in I,j\notin I}\frac{[w_i-w_j-a]}{[w_i-w_j]}\prod_{\substack{i\in I\\j\in\{1,\ldots,n\}}}\frac{[w_i+z_j+a]}{[w_i+z_j]}.$$

  (A similar identity used by Ruijsenaars (2006) to obtain kernel identities in the undeformed case.)

Further source identities

- The last two source identities can be derived as consequences of the Frobenius (1882) determinant evaluation

$$\det_{1\le i,j\le n}\left(\frac{[\lambda+z_i+w_j]}{[\lambda][z_i+w_j]}\right)=\frac{[\lambda+|z|+|w|]\prod_{1\le i<j\le n}[z_i-z_j][w_i-w_j]}{[\lambda]\prod_{1\le i,j\le n}[z_i+w_j]}.$$

Multiple hypergeometric series

The last source identity is at the root of various transformation and summation formulas for multiple elliptic hypergeometric series.

Specifically, consider

$$\phi_N^{m,n}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\middle| \begin{matrix} b_1,\ldots,b_n \\ c_1,\ldots,c_n \end{matrix}\right) = \sum_{\substack{\mu\in\mathbb{N}^m \\ |\mu|=N}} \prod_{1\le i<j\le m} \frac{[x_i - x_j + (\mu_i-\mu_j)\delta]}{[x_i - x_j]}$$

$$\cdot \prod_{i,j=1}^{m} \frac{[x_i - x_j + a_j]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}} \cdot \prod_{i=1}^{m}\prod_{k=1}^{n} \frac{[x_i + b_k]_{\mu_i}}{[x_i + c_k]_{\mu_i}},$$

(where $[z]_k = [z][z+\delta]\cdots[z+(k-1)\delta]$).

Under the balancing condition

$$a_1 + \cdots + a_m = b_1 + \cdots + b_n,$$

Kajihara & Noumi (2003) established

$$\phi_N^{m,n}\left(\begin{matrix} a_1,\ldots,a_m \\ x_1,\ldots,x_m \end{matrix}\middle| \begin{matrix} y_1-b_1,\ldots,y_n-b_n \\ y_1,\ldots,y_n \end{matrix}\right) = \phi_N^{n,m}\left(\begin{matrix} b_1,\ldots,b_n \\ y_1,\ldots,y_n \end{matrix}\middle| \begin{matrix} x_1-a_1,\ldots,x_m-a_m \\ x_1,\ldots,x_m \end{matrix}\right),$$

Multiple hypergeometric series

The trigonometric limit yields the multiple basic hypergeometric series

$$\phi^{m,n}\left(\begin{matrix} a_1,\dots,a_m \\ x_1,\dots,x_m \end{matrix}\middle| \begin{matrix} b_1,\dots,b_n \\ c_1,\dots,c_n \end{matrix}; u\right)$$
$$= \sum_{\mu\in\mathbb{N}^m} u^{|\mu|} \prod_{1\le i<j\le m} \frac{q^{\mu_i}x_i - q^{\mu_j}x_j}{x_i - x_j} \cdot \prod_{i,j=1}^{m} \frac{(a_j x_i/x_j;q)_{\mu_i}}{(q x_i/x_j;q)_{\mu_i}} \cdot \prod_{i=1}^{m}\prod_{k=1}^{n} \frac{(x_i b_k;q)_{\mu_i}}{(x_i c_k;q)_{\mu_i}},$$

(where $(z;q)_k = (1-z)\cdots(1-q^{k-1}z)$).

In particular, it satisfies Kajihara's (2004) far-reaching generalisation of Euler's transformation formula for the Gauss hypergeometric series ${}_2F_1$:

$$\phi^{m,n}\left(\begin{matrix} a_1,\dots,a_m \\ x_1,\dots,x_m \end{matrix}\middle| \begin{matrix} b_1y_1,\dots,b_ny_n \\ cy_1,\dots,cy_n \end{matrix}; u\right)$$
$$= \frac{(\alpha\beta u/c^n;q)_\infty}{(u;q)_\infty} \phi^{n,m}\left(\begin{matrix} c/b_1,\dots,c/b_n \\ y_1,\dots,y_n \end{matrix}\middle| \begin{matrix} cx_1/a_1,\dots,cx_m/a_m \\ cx_1,\dots,cx_m \end{matrix}; \alpha\beta u/c^n\right),$$

with

$$\alpha := a_1\cdots a_m, \quad \beta := b_1\cdots b_n.$$

Thank You!