PERVASIVE FUNCTION SPACES

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Abstract

Let X be a compact Hausdorff topological space and $C(X,\mathbb{C})$ (respectively, $C(X,\mathbb{R})$) the Banach algebra of all continuous complex-valued (respectively, real-valued) functions on X endowed with the uniform norm. A *function space* S on X is a closed subspace of $C(X,\mathbb{C})$. We denote by $\mathrm{clos}_{C(E,\mathbb{C})}\ S$ the closure in $C(E,\mathbb{C})$ of the function space S, where E is a closed subset of X. Similarly, we denote by $\mathrm{clos}_{C(E,\mathbb{R})}\ S$ the closure in $C(E,\mathbb{R})$ of the *real subspace* S of $C(X,\mathbb{R})$.

A function space S of $C(X,\mathbb{C})$ is said to be *complex pervasive* if $\mathrm{clos}_{C(E,\mathbb{C})}\ S = C(E,\mathbb{C})$ whenever E is a proper non-empty closed subset of X. Similarly, a real subspace S of $C(X,\mathbb{R})$ is said to be *real pervasive* if $\mathrm{clos}_{C(E,\mathbb{R})}\ S = C(E,\mathbb{R})$.

In this talk we review the history of this concept, and results to date.

The term pervasive was introduced by Hoffman and Singer in 1960. They studied (complex) pervasive uniform algebras, motivated by the relationship with maximal uniform algebras. They obtained some preliminary results about algebras of analytic functions on connected open subsets of the Riemann sphere.

In 1971, work by Gamelin and Garnett on the so-called Dirichlet algebras revealed connections with analytic capacity, and with the abstract Farrell-Rubel-Shields Theorem.

The real pervasiveness of spaces of harmonic functions on Euclidean spaces was studied by Netuka in 1987. He showed that if the open set $U \in \mathbb{R}^d$ is bounded and connected, and bdy $U =$ bdy clos U, then the space of functions continuous on clos U and harmonic on U is real pervasive on bdy U.

More recent work by the speaker, Netuka and Sanabria has led to rather satisfactory results in spaces of analytic functions of one complex variable (and, separately, spaces of real parts of such functions) on arbitrary open sets and on Riemann surfaces. The results are quite deep, relying on the theory of uniform albebras as well as Vitushkin-type techniques using capacities.