SPECIAL TOPICS IN FUNCTIONAL ANALYSIS THE SPECTRUM OF THE LAPLACIAN, 7.5 HP

JULIE ROWLETT

1. Course logistics

- **Dates:** March 21–April 8, and May 2–June 3, 2016. (There may be virtual meetings April 10–29).
- **Last day for application:** March 21, 2016
- **Examiner and email address for applications:** Julie Rowlett, julie.rowlett@chalmers.se
- **Responsible department:** Mathematics Department

2. Course description

The purpose of this course is to develop fundamental properties of the Laplace operator and its spectrum, culminating in the proof of Weyl's Law. The Laplace operator is one of the most important partial differential operators, and its study is a model case for the general study of elliptic operators. Moreover, to prove even the most basic properties of this operator and its spectrum, as we shall do here, requires quite a lot of mathematical machinery. In the process of developing this machinery, participants will be introduced to the broader fields of geometric and global analysis, which have many research directions and open problems. After this course, you will be able to understand some such directions and current areas of research as well as open problems.

Some of the topics we shall cover include, but are not limited to:

(1) Hilbert and Banach spaces, self-adjoint operators, compact operators, Fredholm operators;
(2) The spectrum of a compact operator, the spectral theorem for compact operators;
(3) The spectral theorem for unbounded operators;
(4) Sobolev spaces, and mapping properties of the Euclidean Laplacian on domains in $\mathbb{R}^n$;
(5) The Laplacian on a Riemannian manifold;
(6) Geometric proof of Weyl's law for the Laplacian on domains in $\mathbb{R}^n$ and compact Riemannian manifolds;
(7) Functional analytic proof of Weyl's law.

Although this course is primarily intended for mathematicians, there is physical motivation underlying the study of the Laplacian and its spectrum. Theoretical physics students are encouraged to participate! The physical derivation of the Laplace equation comes from separating the time and space variables in the wave equation. The idea is then, one has two sides of an equality, one side depending only on time, and the other side depending only on space. Therefore, both sides must be constant. The equation for the space variables is the Laplace equation, and the numbers (which determine the possible values of the aforementioned constant) for which there exists a solution to this equation are the eigenvalues of the Laplace operator. The entirety of these eigenvalues is the spectrum of the Laplacian. The eigenvalues and eigenfunctions of the Laplacian thereby define the fundamental solution of the wave equation, and it turns out they also define the fundamental solution of the heat equation. Although it is not obvious, it is possible to use the spectrum and its properties to define the determinant of the Laplacian, known as the zeta-regularized determinant. This determinant has applications in conformal field theory [8] and Feynmann path integrals [7].

3. Course plan

We shall have eight weeks structured as follows.

3.1. **Functional Analysis Review.**

3.2. **Compact and Fredholm operators.**

3.3. The spectral theorem for compact operators.

3.4. The spectral theorem for unbounded operators.

3.5. The Euclidean Laplacian.

3.6. Weyl's law part I.

3.7. The Laplacian on a Riemannian manifold.

3.8. Weyl's law revisited.

4. The spectrum of the Laplacian, 7.5 hp, Official Information

(1) **Confirmation:** The syllabus was confirmed by the Head of the Department of XXX 200X-XX-XX, 200X-XX- XX.
Disciplinary domain: Science
Department in charge: Department of Mathematical Sciences.
Main field of study: Mathematics
(2) **Position in the educational system:** Elective course; third-cycle education.
(3) **Entry requirements:** Standard real and complex analysis courses at the undergraduate level, and a first course in functional analysis.
(4) **Course content:** In this course we will: (1) introduce the spectrum of an operator acting on a Hilbert space; (2) prove the spectral theorem for both compact operators and unbounded operators; (3) introduce the Euclidean Laplacian and the Laplacian on a compact Riemannian manifold and develop its properties as an operator acting on a suitable Hilbert space; (4) prove Weyl's law for the asymptotic behavior of the spectrum; (5) connect these topics to current research areas and physics to as great of an extent as is possible.
(5) **Outcomes:** After the completion of this course the Ph.D. student is expected to be able to:
- Outline the major points in the proofs of the spectral theorems;
- Present the functional analytic properties of the Laplacian;
- Outline at least one proof of Weyl's law.

(6) **Required reading:** During the course lecture notes will be compiled in a collective process involving both the students and the examiner. Supplementary reading material is listed in the bibliography below.
(7) **Assessment:** At the end of the course, students will be free to choose between two options:
(a) an oral exam, or
(b) a written take-home exam.

A Ph.D. student who has failed a test twice has the right to change examiners, if it is possible. A written application should be sent to the Department.

In cases where a course has been discontinued or major changes have been made a Ph.D. should be guaranteed at least three examination occasions (including the ordinary examination occasion) during a time of at least one year from the last time the course was given.

(8) **Grading scale:** The grading scale consists of Pass (Godkänd) or Fail (U).
(9) **Course evaluation:** The course evaluation is carried out together with the Ph.D. students at the end of the course, and is followed by an individual, anonymous survey. The results and possible changes in the course will be shared with the students who participated in the evaluation and to those who are beginning the course.
(10) **Language of instruction:** The language of instruction is English.

References

[1] Isaac Chavel, *Eigenvalues in Riemannian Geometry.*
[2] H. Brezis, *Analyse fonctionnelle, Théorie et applications.*
[3] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleicuhungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung),* Math. Ann. 71, no. 4, (1912), 441–479.
[4] S. Rosenberg, *The Laplacian on a Riemannian Manifold.*
[5] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol. 1.*
[6] F. Hirzebruch and W. Scharlau, *Einführung in die Funktionalanalysis.*

[7] S.W. Hawking, *Zeta function regularization of path integrals in curved spacetime,* Comm. Math. Phys. 55, no. 2, (1977), 133–148.
[8] A. M. Polyakov, *Quantum geometry of bosonic strings,* Physics Letters B, vol. 103, no. 3, (1981), 207–210.