Opinion dynamics

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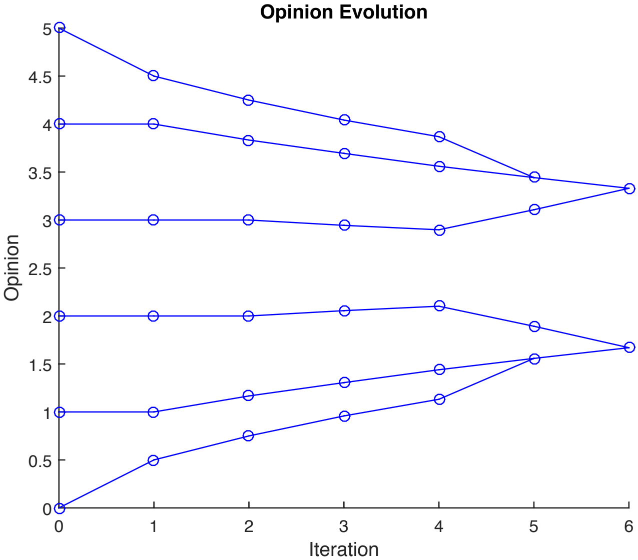
Opinion Evolution
Opinion
Iteration
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
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0
1
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5
6

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$$x_i(t+1) = \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{N}_i(t)} x_j(t),$$

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- The dynamics are unaffected by rescaling (update rule is linear), so WLOG $r = 1$.

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The most famous model which incorporates #1 but neither #2 nor #3 is the **Deffuant-Weisbuch** model.

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$$\eta_{t+}(u) = \eta_{t-}(u) + \mu(\eta_{t-}(v) - \eta_{t-}(u)),$$
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  Case 2: Otherwise, $(\eta_{t+}(u), \eta_{t+}(v)) = (\eta_{t-}(u), \eta_{t-}(v))$.

The **energy** of a Hegselmann-Krause system $\mathbf{x} = (x_1, \dots, x_n)$ is given by

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$$\text{Alt. 2: } \mathcal{E}(\mathbf{x}(t)) - \mathcal{E}(\mathbf{x}(t+1)) \geq (1 - \lambda_t^2)\mathcal{E}_{\text{active}}(\mathbf{x}(t)),$$

where

$$\mathcal{E}_{\text{active}}(\mathbf{x}) = \sum_{i \sim j} ||x_i - x_j||^2,$$

$\lambda_t = \max\{|\lambda| : \lambda \neq 1$ is an eigenvalue of P_t, where $\mathbf{x}_{t+1} = P_t \mathbf{x}_t.\}$

- Using the "standard estimate"

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Regarding **lower bounds** on the freezing time in Euclidean space ...

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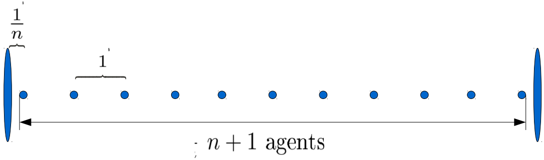

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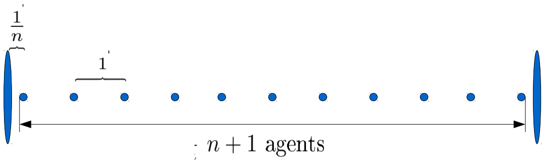

Figure: The "dumbbell" configuration $\mathcal{D}_n$. Each dumbbell has weight n.

Proof relates the time evolution of this configuration to properties of a certain random walk on a path graph.

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In $\mathbb{R}^1$, an additional complication is that, in contrast to the homogeneous case, agents can *cross*.

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- **Open Problem 3:** Is the evolution of every semi-infinite sequence of equally spaced opinions *ultimately* periodic ?

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To get started: In $\mathbb{R}^1$ is there a critical length L_c for a.a.s. consensus ?

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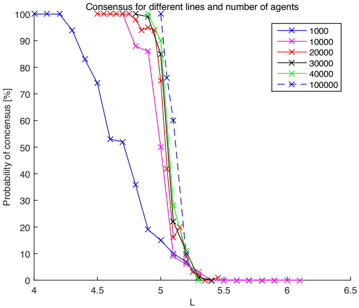

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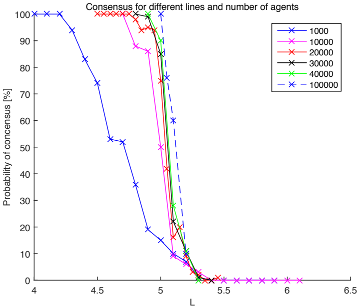

- Equally spaced configurations are easier to simulate. As the inter-agent spacing $d \to 0$, it seems that the diameter of the first cluster to break off tends to a limit of around 2.38.

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- Also **open** what happens at $\theta = 1/2$.

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- Cannot prove the existence of a critical L_c this way, however. As $L \uparrow L_c$, typically *semistable* configurations appear which take a long time to freeze. Thus, one loses control over the approximation errors.
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What we think we know for one-dimensional HK:

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- However, the most intriguing finding is that it seems that consensus reappears for value of $L \approx 6$, and we can prove a.a.s. consensus for some L-values in this region.
  $\Rightarrow$ There seems to be a triple phase-transition !