The Hegselmann-Krause model of opinion dynamics

Peter Hegarty
(with the help of: Edvin Wedin, Anders Martinsson, Mattias Danielsson, Jimmie Ekström, Jesper Johansson and Gustav Karlsson)

Department of Mathematics, Chalmers/Gothenburg University

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Work in Progress: Typical behaviour of random configurations

A beautiful result is that, if $G = \mathbb{Z}$ and initial opinions are i.i.d. in $[0, 1]$, then

(i) If $\theta > 1/2$ then, for any $\mu$, almost surely all opinions converge to $1/2$.

(ii) If $\theta < 1/2$ then, for any $\mu$, almost surely disagreement persists.
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- At discrete time steps, a random pair of neighbors in $G$ “meet and discuss”.

Parameters $\theta \in [0, 1]$ and $\mu \in (0, 1/2]$ such that, if agents with opinions $(a, b)$ meet, then afterwards their opinions will have changed to

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\begin{cases} 
(a + \mu(b - a), b - \mu(b - a)), & \text{if } |a - b| \leq \theta, \\
(a, b), & \text{if } |a - b| > \theta.
\end{cases}
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A finite number, $n$, say, of agents, indexed by the integers $1, 2, \ldots, n$. Time is discrete: $t = 0, 1, \ldots$. A real number $x_i(t)$ represents the opinion of agent $i$ at time $t$. There is a confidence bound $r > 0$, which is the same for all agents. Opinions are updated synchronously according to $x_i(t+1) = 1/|N_i(t)| \sum_{j \in N_i(t)} x_j(t)$, where $N_i(t) = \{j: ||x_j(t) - x_i(t)|| \leq r\}$. The dynamics are unaffected by rescaling (update rule is linear), so WLOG $r = 1$. 

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Figure: Evolution for 5 equally spaced agents, initially placed at 0, 1, 2, 3, 4.
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Interpretation: There are $k$ issues, and two agents must be close on all issues for compromise to occur.
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**Example 2.** The circle $V = \mathbb{T}^1$, of diameter greater than 2.
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**Example 2.** The circle $V = \mathbb{T}^1$, of diameter greater than 2.

Interpretation: Imagine, for example, that the issue under discussion is the time of day or year for holding some event.
Convergence in $\mathbb{R}$:

- Very easy to show that opinions converge to limiting values (general nonsense, Banach Fixed Point Theorem blah blah ...)
- In fact quite easy to show that opinions freeze, i.e.: there is always some $T > 0$ such that $x_i(t) = x_i(T)$ for all $i$ and all $t \geq T$.
- Still quite easy to show that the freezing time is bounded by a universal polynomial function of the number of agents: $\Rightarrow$ Can get a bound of around $O(n^5)$ from general theory of Markov chains on graphs.
- $\Rightarrow$ Best to date is $O(n^3)$. Elementary argument which considers the behaviour of the extremal agents (Bhattacharyya et al, 2013).
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- We believe that the freezing time is always \( O(n^2) \), but this remains open.
Figure: Schematic representation of the configuration $\mathcal{D}_n$. Each dumbbell has weight $n$. 
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The **energy** of a Hegselmann-Krause system $\mathbf{x} = (x_1, \ldots, x_n)$ is given by

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Basic Result: The dynamics always decrease the energy.

$$
\mathcal{E}(x(t)) - \mathcal{E}(x(t + 1)) \geq 4 \cdot \sum_{i=1}^{n} \|x_i(t) - x_i(t + 1)\|^2.
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- For lower bounds, $n$ agents placed equidistantly around a circle will also require time $\Omega(n^2)$ to freeze. This is a genuinely 2-dimensional example. Also, in contrast to the dumbbell, this configuration reaches consensus.
- We believe that the freezing time is $O(n^2)$ in all dimensions.
Convergence on $\mathbb{T}^1$: In contrast to the Euclidean case, configurations no longer need to freeze in finite time. Moreover, in a frozen configuration, no cluster need be isolated. E.g.: agents spaced equally around the circle at distance one. However, there are even non-periodic frozen configurations. Hendrickx et al (2009) asked if opinions must always converge on the circle. Proven by us [HMW]. Proof uses both the energy reduction technique and a modification of the idea in Bhattachrya et al, both suitably modified for the circle. The influence digraph can change at most $O(n^4)$ times. However, it can take arbitrarily long for these changes to occur. Can also prove convergence in $\mathbb{T}^k$ for all $k \geq 1$ (technical).
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**Nothing** is yet proven. Indeed, evidence against monotonicity is the fact that increasing the confidence bound $r$ can sometimes destroy consensus.
Simulations:

Work in Progress: Typical behaviour of random configurations

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Many simulations performed in [DEJK, 2015] for uniform distributions of agents in regions of $\mathbb{R}^1$ and $\mathbb{R}^2$.

In $\mathbb{R}^1$ there is only one “region”, namely an interval. Simulations give evidence for existence of a critical length, slightly above 5.

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**Idea 1:** Go to the limit of a continuum of agents.

**Idea 2:** Study configurations of equally spaced agents.
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$$x_{t+1}(\alpha) = \frac{1}{\mu(N_t(\alpha))} \int_{N_t(\alpha)} x_t(\beta) \, d\beta,$$

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- **Approximation between DAM and CAM:** Hendrickx et al (2009) have results which are probably strong enough for most purposes.
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- **Approximation between DAM and CAM:** Hendrickx et al (2009) have results which are probably strong enough for most purposes. So it remains to study the CAM-model.
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Problem remains open for linear functions (those corresponding to a uniform distribution of opinions).
Idea 2: Equally spaced agents

Recall that in [HW] we proved that a configuration of \( n \) agents, with initial opinions \( 0, 1, \ldots, n-1 \), evolves periodically, with groups of 3 agents breaking loose at each end every 5th time step.

It is conceptually easier to consider a semi-infinite configuration of equally spaced agents, with initial opinions at all non-negative integers. The first (and main) step in [HW] was to prove that this configuration evolves periodically, with a group of 3 agents breaking loose on the left after every 5th time step.

Now one should consider a general inter-agent spacing \( d \in (0, 1] \) - ultimately we are interested in letting \( d \to 0 \).
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Most intriguingly, simulations suggest a possible triple phase transition!