Can connected commuting graphs of finite groups have arbitrarily large diameter?

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We present a way to construct a family of random groups related to the conjecture of Iranmanesh and Jafarzadeh about commuting graphs of finite groups. Let G be a non-abelian group. We define the *commuting graph* of G, denoted by $\Gamma(G)$, as the graph whose vertices are the non-central elements of G, and such that $\{x, y\}$ is an edge if and only if $xy = yx$. One can just as well define the graph to have as its vertices the non-identity cosets of $Z(G)$, with $\{Zx, Zy\}$ adjacent if and only if $xy = yx$ and we stick to this definition henceforth. The conjecture of Iranmanesh and Jafarzadeh is as follows.

Conjecture 1. (Iranmanesh and Jafarzadeh, [5]) There is a natural number b such that if G is a finite, non-abelian group with $\Gamma(G)$ connected, then $\text{diam}(\Gamma(G)) \leq b$.

The initial motivation was to show that Conjecture 1 is false by providing a counterexample using probabilistic methods. Some partial results in favor of Conjecture 1 (see details in the full length version of the present note, [4]) were already known at the moment the work on this project was initiated. It might seem natural to guess that for the commuting graph to be of large diameter, the group itself should be far from being abelian. However, it turns out in many cases the opposite holds and the commuting graph is connected and is of small diameter. It is thus reasonable to look at "more abelian" groups. Guidici and Pope [3] were first to consider the case of p-groups and provided a few notable results in support of Conjecture .

Let us recall some basic definitions first. If x, y are two elements of a group G, then their *commutator* $[x, y]$ is defined to be the group element $x^{-1}y^{-1}xy$. The commutator subgroup of G is the subgroup generated by all the commutators and is denoted G'. If $G' \subseteq Z(G)$ one says that G is of *nilpotence class* 2. Quite surprisingly, one of the results of Guidici and Pope was that in this case the centre of the group should be of considerable size, otherwise the conjecture holds.

Theorem 2. *If G is of nilpotence class 2 and $|Z(G)|^3 < |G|$, then* $diam(\Gamma(G)) = 2$.

The general idea behind our construction is that if Conjecture 1 is false, then it should already fail among groups of nilpotence class two. Even more, one can take G such that both $Z(G)$ and $G/Z(G)$ are both elementary abelian 2-groups, that is, additive groups of some vector spaces over $\mathbb{F}_2$. However, instead of trying to construct an explicit counterexample we are going to introduce randomness in defining commutator relations in order to study how the commuting graph of a typical group of that kind looks like. As illustrated by many applications of the probabilistic method pioneered by Erdős (see [1] for the full treatment), the behaviour of a random object is often easer to analyse, so by adjusting parameters it is sometimes possible to provide an example with desired properties. Unfortunately, we were unable to disprove the conjecture in full in this way, but were able to produce a group whose commuting graph is of diameter 10, which became the largest value achieved by that time[1].

Before we proceed with the model of random groups, let us describe the significant success which took place since our work was undertaken. In [2], Giudici and Parker provide explicit examples of connected commuting graphs of unbounded diameter, thus disproving Conjecture 1. Their construction is based on and inspired by the random groups presented here, though they were able to devise an explicit construction. They have checked by computer that their model produces examples of commuting graphs of every diameter between 3 and 15, though it appears to remain open whether every positive integer diameter is achievable. As a remarkable counterpoint to their result, Morgan and Parker [6] have proved that if G has trivial centre then every connected component of $\Gamma(G)$ has diameter at most 10. Note that this condition specifically excludes nilpotent groups. In contrast to these purely group-theoretical advances, we are not aware of any further progress having been made on the analysis of the random groups described below.

Returning to our random construction, the group is defined as follows. Let m, r be positive integers and $V = V_m$ and $H = H_r$ be vector spaces over $\mathbb{F}_2$ of dimensions m and r respectively. Let $\phi : V \times V \to H$ be a bilinear map. Set $G := V \times H$ and define a multiplication on G by

$$(v_1, h_1) \cdot (v_2, h_2) := (v_1 + v_2, h_1 + h_2 + \phi(v_1, v_2)). \tag{1}$$

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Then it is easy to check that

(i) $(G, \cdot)$ is a group of order 2^{m+r}, with identity element $(0, 0)$.
(ii) Let $\mathcal{H} := \{(0, h) : h \in H\}$. Then $\mathcal{H}$ is a subgroup of G and $G/\mathcal{H} \cong V$, as an abelian group.
(iii) $G' \subseteq \mathcal{H} \subseteq Z(G)$.
(iv) G is abelian if and only if ϕ is symmetric.
(v) The commutator of two elements is given by

$$[(v_1, h_1), (v_2, h_2)] = (0, \phi(v_1, v_2) - \phi(v_2, v_1)) \tag{2}$$

The map $\phi(\cdot, \cdot)$ is taken uniformly at random among all possible bilinear maps. It is then clear, due to (2), that, for two fixed distinct elements of G, their commutator becomes uniformly distributed on $\mathcal{H}$. Moreover, if we fix a basis $(v_1, ..., v_m)$ of V then all the commutator relations are determined by the skew-symmetric matrix A with $A_{i,j} = \phi(v_i, v_j) - \phi(v_j, v_i)$. Now we are going to define the parameters m and r such that the commuting graph $\Gamma(G)$ is similar to the Erdős–Rényi graph $G_{n,p}$ with $p = n^{-1+\epsilon}$, which is known to have diameter concentrated at $\lceil 1/\epsilon \rceil$ with high probability for small $\epsilon > 0$.

Let $k \geq 2$ be an integer, and $\delta \in \left(0, \frac{1}{2k(k-1)}\right)$ a real number. There is a choice of real number $\delta_1 > 0$ such that the following holds: for each positive integer m, if we set

$$r := \lfloor (1 - \delta_1)m \rfloor, \quad p := 2^{-r}, \quad n := 2^m - 1, \tag{3}$$

then, for all m sufficiently large,

$$1 + \log_n p \in \left(\frac{1}{k} + \delta, \frac{1}{k-1} - \delta\right). \tag{4}$$

The probability that an edge of $\Gamma(G)$ is present is then p, as this is the probability that a uniformly chosen random element of $\mathcal{H}$ is zero. Thus one can hope that its diameter is concentrated around k, as it would be if the states of all edges were independent as in $G_{n,p}$.

Unfortunately, it becomes difficult to translate the known methods of $G_{n,p}$ to our setting due to large amount of dependence between edges, so we were unable to prove this correspondence in full. However, some convincing structural results appear to be amenable to the second moment method.

Proposition 3. *Let $G_{m,k}$ be the group defined above with corresponding parameters m, r and k. Then*

(i) *As $m \to \infty$, $\mathbb{P}(G' = Z(G) = \mathcal{H}) \to 1$.*
(ii) *There is some $\delta_3 > 0$, depending on the choices of δ and δ_1, such that, as $m \to \infty$, $\Gamma(G_{m,k})$ almost surely has a connected component of size at least $n - n^{1-\delta_3}$. The diameter of $\Gamma(G_{m,k})$ is at least k w.h.p., but might be infinite if it is not connected.*

So in fact to provide a counterexample to Conjecture it is sufficient to prove that $\Gamma(G_{m,k})$ remains connected for large m and fixed k. We conjecture that even a more precise statement holds.

Conjecture 4. As $m \to \infty$, $\Gamma(G_{m,k})$ is almost surely connected and of diameter k.

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