Solutions to extra problems (28/12/05)

1. The volume of the cylinder is $\pi r^2 h$. Tha area of the curved portion is $2\pi r h$ and each of the top and bottom has area πr^2, which means that the total surface area is $2\pi r h + 2\pi r^2$. The assumption in this exercise is thus that

$$\pi r^2 h = 2 \times [2\pi r h + 2\pi r^2].$$

After a little algebra, this reduces to

$$h = \frac{4r}{r-4}. \tag{1}$$

From this it follows immediately that r must be greater than 4, as h is a priori positive. If $r = 6$ then (1) gives that $h = 12$ and thus the volume in this case is $\pi \times 6^2 \times 12 = 432\pi$ cubic metres.

2. On the one hand,

$$\frac{\text{Volume of B}}{\text{Volume of A}} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}.$$

On the other hand,

$$\frac{\text{Volume of C}}{\text{Volume of A}} = \frac{1}{2} \times \left(\frac{4}{5}\right)^2 = \frac{8}{25}.$$

Hence, C has the larger volume and its volume is 27/25 times that of B.

3 (i) The desired fraction is simply equal to

$$\frac{\text{Area of a circle of radius 1}}{\text{Area of a square of radius 1}} = \frac{\pi}{4}.$$

(ii) As is 'intuitively clear', the extra circle should be placed so that its centre coincides with the point of contact of two of the existing circles. The newly covered area - call it $\mathcal{X}$ - then consists of two equal parts A and B. Each of these is in turn of the form $C - 2 \cdot D$, where C is a 60-degree circular section and D is the area between a circle and one side of a regular hexagon inscribed in it. Thus, the area we're looking for is

$$\mathcal{X} = 2 \cdot [\text{Area(C)} - 2 \cdot \text{Area(D)}]. \tag{2}$$

First of all, we have

$$\text{Area(C)} = r\theta = 1 \cdot \frac{\pi}{3} = \frac{\pi}{3}. \tag{3}$$

Next, $D = \frac{1}{6} \cdot (E - F)$, where E is the whole circle and F the inscribed hexagon. We have Area(E) $= \pi r^2 = \pi \cdot 1^2 = \pi$. The area F was computed in exercise **5(iii)** in the exercises on 'cirklar och vinklar'. Thus Area(F) $= \frac{3\sqrt{3}}{2}$. We conclude that

$$\text{Area(D)} = \frac{1}{6} \cdot \left(\pi - \frac{3\sqrt{3}}{2} \right). \tag{4}$$

Substituting (3) and (4) into (2) and simplifying, we find that $\mathcal{X} = \sqrt{3}$. Thus $\sqrt{3}$ is the maximum extra area which can be covered by the addition of one more circle.

4 (i) Divide the octagon into 8 congruent triangles in the obvious manner. Each triangle has base 1 and perpendicular height $\sin 45 = 1/\sqrt{2}$. Hence the area of each triangle is $\frac{1}{2\sqrt{2}}$, so that the area of the octagon is $\frac{4}{\sqrt{2}}$.
(ii) We need to compute the length of the third side in each of the above triangles. Each of the other sides has length 1 and these enclose an angle of 45 degrees. Draw a perpendicular height to one of these two sides. Its length is $\sin 45 = 1/\sqrt{2}$ and it cuts off a length $\cos 45 = 1/\sqrt{2}$ from the base. An application of Pythagoras theorem then implies that the length of the third side is

$$\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2} = \sqrt{2 - \sqrt{2}}.$$

Hence, the circumference of the octagon is $8\sqrt{2-\sqrt{2}}$, which is less than that of the circle, namely 2π. Thus we achieve the lower bound

$$\pi > 4\sqrt{2 - \sqrt{2}} = 3.0614....$$

5. Two applications of Pythagoras theorem (the first of which gives the length of a diagonal in the base of the box) give that the distance between opposite corners is

$$\sqrt{4^2 + 6^2 + 7^2} = \sqrt{101} \ \text{ metres.}$$

6 (i) The lengths of the other sides are $10\frac{1}{2}$ and $12\frac{1}{4}$ metres.

(ii) The ratio of the new to the old surface area is $\left(\frac{7}{4}\right)^2 = \frac{49}{16} = 3\frac{1}{16}$.

(iii) The ratio of the new to the old volume is $\left(\frac{7}{4}\right)^3 = \frac{343}{64} = 5\frac{23}{64}$.

7. Let r denote the radius of the cone. From the appropriate pair of similar triangles, we obtain the relation

$$\frac{r}{5} = \frac{20}{20-8} = \frac{20}{12},$$

whence $r = \frac{25}{3}$. Hence the volume of the cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{25}{3}\right)^2 \times 20 = \frac{12500}{27}\pi$ cubic metres.

8. Let d be the distance which Jane must move back in part **(i)** and let h be the height of the building. Then from the similarity of three appropriately chosen triangles, we obtain the relations

$$\frac{h}{2040+d} = \frac{5}{40+d} = \frac{1,84}{d}.$$

Equaltiy of the second and third expressions leads to the result (after the usual algebra) that $d \approx 23,29$ metres. Then with the help of the first expression, we deduce that $h \approx 163$ metres.