2-PILE NIM WITH A RESTRICTED NUMBER OF MOVE-SIZE IMITATIONS

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ABSTRACT. We study a variation of the combinatorial game of 2-pile Nim. Move as in 2-pile Nim but with the following constraint:

Suppose the previous player has just removed say $x > 0$ tokens from the shorter pile (either pile in case they have the same height). If the next player now removes x tokens from the larger pile, then he imitates his opponent. For a predetermined natural number p, by the rules of the game, neither player is allowed to imitate his opponent on more than $p - 1$ consecutive moves.

We prove that the strategy of this game resembles closely that of a variant of Wythoff Nim—a variant with a blocking manoeuvre on $p - 1$ diagonal positions. In fact, we show a slightly more general result in which we have relaxed the notion of what an imitation is.

WITH AN APPENDIX BY PETER HEGARTY

1. INTRODUCTION

A finite impartial game is usually a game where

- there are 2 players and a starting position,
- there is a finite set of possible positions of the game,
- there is no hidden information,
- there is no chance-device affecting how the players move,
- the players move alternately and obey the same game rules,
- there is at least one final position, from which a player cannot move, which determines the winner of the game and
- the game ends in a finite number of moves, no matter how it is played.

If the winner of the game is the player who makes the final move, then we play under normal play rules, otherwise we play a misère version of the game.

In this paper *a game*, say G, is always a finite impartial game played under normal rules. The player who made the most recent move will be denoted by *the previous player*. A position from which the previous player will win, given best play, is called a *P-position*, or just P. A position from which the *next player* will win is called an *N-position*, or just N. The set of all P-positions will be denoted by $\mathcal{P} = \mathcal{P}_G$ and the set of all N-positions by $\mathcal{N} = \mathcal{N}_G$.

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Suppose A and B are the two piles of a 2-pile take-away game, which contain $a \geq 0$ and $b \geq 0$ tokens respectively. Then the *position* is (a, b) and a *move* (or an *option*) is denoted by $(a, b) \to (c, d)$, where $a - c \geq 0$ and $b - d \geq 0$ but not both $a = c$ and $b = d$. All our games are symmetric in the sense that (a, b) is P if and only if (b, a) is P. Hence, to simplify notation, when we say (a, b) is P (N) we also mean (b, a) is P (N). Througout this paper, we let $\mathbb{N}_0$ denote the non-negative integers and $\mathbb{N}$ the positive integers. For integers $a < b$ we let $[a, b]$ denote the set $\{a, a + 1, \ldots, b\}$.

1.1. **The game of Nim.** The classical game of Nim is played on a positive number of piles, each containing a non-negative number of tokens, where the players alternately remove tokens from precisely one of the non-empty piles—that is, at least one token and at most the entire pile—until all piles are gone. The winning strategy of Nim is, whenever possible, to move so that the "Nim-sum" of the pile-heights equals zero, see for example [Bou] or [SmSt] (page 3). When played on one single pile there are only next player winning positions except when the pile is empty. When played on two piles, the pile-heights should be equal to ensure victory for the previous player.

1.2. **Adjoin the P-positions as moves.** A possible extension of a game is $(\star)$ *to adjoin the P-positions of the original game as moves in the new game.* Clearly this will alter the P-positions of the original game.

Indeed, if we adjoin the P-positions of 2-pile Nim as moves, then we get another famous game, namely Wythoff Nim (a.k.a Corner the queen), see [Wy]. The set of moves are: Remove any number of tokens from one of the piles, or remove the same number of tokens from both piles.

The P-positions of this game are well-known. Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the golden ratio. Then (x, y) is a P-position if and only if

$$(x, y) \in \left\{ \left(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor\right) \mid n \in \mathbb{N}_0 \right\}.$$

We will, in a generalised form, return to the nice arithmetic properties of this and other sequences in Proposition 1 (see also [HeLa] for further generalisations).

Other examples of $(\star)$ are the Wythoff-extensions of n-pile Nim for $n \geq 3$ discussed in [BlFr, FrKr, Su, SuZe] as well as some extensions to the game of 2-pile Wythoff Nim in [FraOz], where the authors adjoin subsets of the Wythoff Nim P-positions as moves in new games.

1.3. **Remove a game's winning strategy.** There are other ways to construct interesting extensions to Nim on just one or two piles, for example we may introduce a so called *move-size dynamic* restriction, where the options in some specific way depend on how the previous player moved (for example how many tokens he removed), or "pile-size dynamic"[1] restrictions, where the options depend on the number of tokens in the respective piles.

The game of "Fibonacci Nim" in [BeCoGu] (page 483) is a beautiful example of a move-size dynamic game on just one pile. This game has

[1]We understand that pile-size dynamic games are not 'truly' dynamic since for any given position of a game, one may classify each P-position without any knowledge of how the game has been played up to this point.

been generalised, for example in [HoReRu]. Treatments of two-pile move-size dynamic games can be found in [Co], extending the (pile-size dynamic) "Euclid game", and in [HoRe].

The games studied in this paper are move-size dynamic. In fact, similar to the idea in Section 1.2, there is an obvious way to alter the P-positions of a game, namely ($\star\star$) *from the original game, remove the next-player winning strategy.* For 2-pile Nim this means that we remove the possibility to *imitate* the previous player's move, where imitate has the following interpretation:

Definition 0 Given two piles, A and B, where $\#A \leq \#B$—and where the number of tokens in the respective pile is counted before the previous player's removal of tokens—then, if the previous player removed tokens from pile A, the next player *imitates* the previous player's move if he removes the same number of tokens from pile B as the previous player removed from pile A.

This game, we call *Imitation Nim.* The intuition is, given the position (a, b), where $a \leq b$, Alice can prevent Bob from going to (c, d), where $c < a$ and $b - a = d - c$, by moving $(a, b) \to (c, b)$. We illustarate with an example:

Example 1 Suppose the game is Imitation Nim and the position is $(1, 3)$. If this is an a initial position, then there is no 'dynamic' restriction on the next move so that the set $\{(1, 2), (1, 1), (1, 0), (0, 3)\}$ of Nim options is identical to the set of Imitation Nim options. But this holds also, if the previous player's move was

(1) $$(1, x) \to (1, 3),$$

or

(2) $$(x, 3) \to (1, 3),$$

where $x \geq 4$. For these cases, the imitation rule does not apply since the previous player removed tokens from the pile with more tokens.

If on the other hand, the previous move was

(3) $$(x, 3) \to (1, 3),$$

where $x \in \{2, 3\}$ then, by the imitation rule, precisely the option $(1, 3) \to (1, 3 - x + 1)$ is prohibited.

Further, $(3, 3) \to (1, 3)$ is a losing move—since, as we will see in Proposition 0(i), $(1, 3) \to (1, 2)$ is a winning move. But, by the imitation rule, $(2, 3) \to (1, 3)$ is a winning move—since for this case $(1, 3) \to (1, 2)$ is forbidden.

This last observation leads us to ask a general question for a move-size dynamic game, roughly: *When does the move-size dynamic rule change the outcome of a game?* To clarify this question, let us introduce some non-standard terminology, valid for any move-size dynamic game.

Definition 1 Let G be a move-size dynamic game. A position $(x, y) \in G$ is

(1) *dynamic*: if, in the course of the game, we cannot tell whether it is P or N without knowing the history—at least the most recent move—of the game;
(2) *non-dynamic*
P: if it is P regardless of any previous move(s),
N: ditto, but N.

Remark 1 Henceforth, if not stated otherwise, we will think of a (move-size dynamic) game as a game where the progress towards the current position is memorized in an appropriate manner. A consequence of this approach is that each (dynamic) position is P or N.

In light of these definitions, we will now characterize the winning positions of a game of Imitation Nim (see also Figure 1)—this is a special case of our main theorem in Section 2, notice for example the absence of Wythoff Nim P-positions that are dynamic, considered as positions of Imitation Nim.

Proposition 0 Let $0 \leq a \leq b$ be integers. Suppose the game is Imitation Nim. Then (a, b) is

(i) non-dynamic P if and only if it is a P-position as of Wythoff Nim;
(ii) non-dynamic N if and only if
  (a) there are integers $0 \leq c \leq d < b$ with $b - a = d - c$ such that (c, d) is a P-position of Wythoff Nim, or
  (b) there is a $0 \leq c < a$ such that (a, c) is a P-position of Wythoff Nim.

Remark 2 Given the notation in Proposition 0, it is wellknown (see also Figure 1) that: There is an $x < a$ such that (x, b) is a P-position of Wythoff Nim implies (iia). One may also note that, by symmetry, there is an intersection of type (iia) and (iib) positions, namely whenever $a = d$, that is whenever $c < a < b$ is an arithmetic progression.

By Proposition 0 and Remark 2 (c, b) is a dynamic position of Imitation Nim if and only if there is a P-position of Wythoff Nim, (c, d), with $c \leq d < b$. Further, with notation as in (iia), we get that (c, b) is dynamic P if and only if the previous player moved $(a, b) \to (c, b)$.

Recall that the first few P-positions of Wythoff Nim are

$$(0, 0), (1, 2), (3, 5), \ldots.$$

Hence, in Example 1, a (non-dynamic) P-position of Imitation Nim is $(1, 2)$. The position $(1, 3)$ is, by the comment after Remark 2, dynamic. The positions $(2, 3)$ and $(3, 4)$ are non-dynamic N-positions, by Proposition 0(iia). As examples of non-dynamic N-positions of type (iib), we may take $(2, x)$, $x \geq 3$. By the comment after Remark 2 (again), we get:

Corollary 0 Treated as initial positions, the P-positions of Imitation Nim are identical to the P-positions of Wythoff Nim.

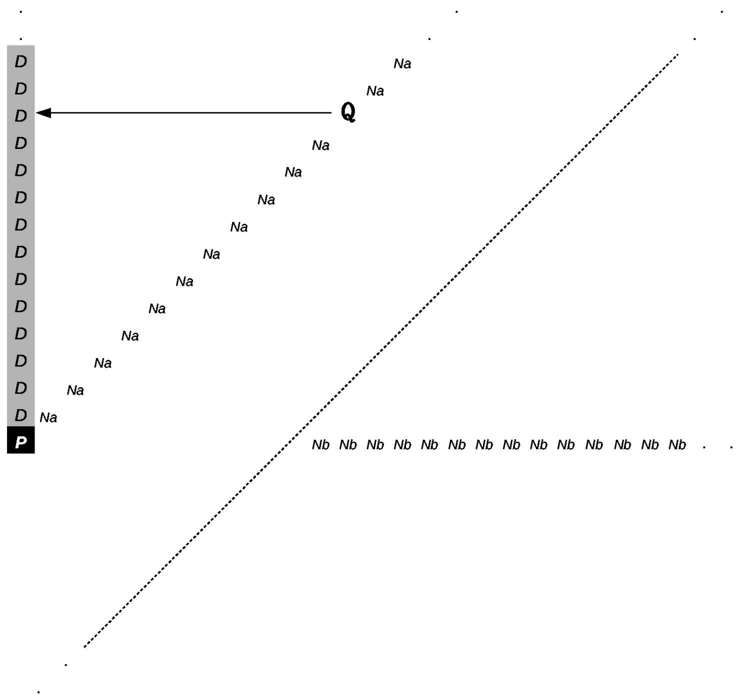

FIGURE 1. The strategy of Imitation Nim. The P is a (Wythoff Nim) P-position north of the main diagonal. The D:s are dynamic positions. The arrow symbolises a winning move from Q. The Na:s are the positions of type (iia) in Proposition 0, the Nb:s of type (iib).

Remark 3 For a given position, the rules of Wythoff Nim allow more options than those of Nim, whereas the rules of Imitation Nim give fewer. Nevertheless, the P-positions are identical if one only considers starting positions. Hence, one might want to view these variants of 2-pile Nim as each others "duals".

1.4. **Two extensions of Imitation Nim and their "duals".** Our first reference for a move-size dynamic game is [BeCoGu]. But we have not been able to find any literature on the subject of *games with memory*, which is our next topic.

1.4.1. *A game with memory.* A natural extension of Imitation Nim is, given $p \in \mathbb{N}$, to allow $p-1$ consecutive imitations—by one and the same player—but to prohibit the p:th imitation. We denote this game by $(p, 1)$-*Imitation Nim.*

Remark 4 This rule removes the winning strategy from 2-pile Nim if and only if the number of tokens in each pile is $\geq p$.

Example 2 Suppose the game is $(2, 1)$-Imitation Nim (so that no two consecutive imitations by one and the same player are allowed). Suppose the starting position is $(2, 2)$ and that Alice moves to $(1, 2)$. Then, if Bob moves to $(1, 1)$, Alice will move to $(0, 1)$, which is P for a game with this particular history. This is because the move $(0, 1) \to (0, 0)$ would have been a second consecutive imitation for Bob and hence is no option. If Bob chooses instead to move to $(0, 2)$ then Alice can win in the next move, since $2 > 1$ (so the imitation rule does not apply).

Indeed, Alice's first move is a winning move, so $(2, 2)$ is N (which is non-dynamic) and $(1, 2)$ is P. But, if $(1, 2)$ would have been an initial position then it would have been N, since $(1, 2) \to (1, 1)$ would have been a winning move. So $(1, 2)$ is dynamic.

Clearly $(0, 0)$ is non-dynamic P. Otherwise the 'least' non-dynamic P-position is $(2, 3)$, since $(2, 2)$ is N and $(2, 1)$ or $(1, 3) \to (1, 1)$ would be winning moves, as would $(2, 0)$ or $(0, 3) \to (0, 0)$.

1.4.2. *The dual of (p,1)-Imitation Nim.* In [HeLa, Lar] we put a *Muller twist* or *blocking manoeuvre* on the game of Wythoff Nim. For an introduction to the concept of a blocking manoeuvre, see for example [SmSt]. Variations on Nim with a Muller twist can also be found, for example, in [GaSt] (which generalises a result in [SmSt]), [HoRe1] and [Zh].

Fix two positive integers p and m. Suppose the current pile-position is (a, b). The rules are: Before the next player removes any tokens, the previous player is allowed to announce $j \in [1, p-1]$ positions, say $(a_1, b_1), \ldots, (a_j, b_j)$ where $b_i - a_i = b - a$, to which the next player may not move. Once the next player has moved, any blocking manoeuvre is forgotten. Otherwise move as in Wythoff Nim.

We will show that as a generalisation of Corollary 0, if X is a starting position of $(p, 1)$-Imitation Nim then it is P if and only if it is a P-position of $(p, 1)$-Wythoff Nim. Further, a generalisation of Proposition 0 holds, but let us now move on to our next extension of Imitation Nim.

1.4.3. *A relaxed imitation.* Let $m \in \mathbb{N}$. We relax the notion of an imitation to an *m-imitation* (or just imitation) by saying: provided the previous player removed x tokens from pile A, with notation as in Definition 0, then the next player m-imitates the previous player's move if he removes $y \in [x, x+m-1]$ tokens from pile B.

Definition 2 Let $p \in \mathbb{N}$. We denote by *(p, m)-Imitation Nim* the game where no p consecutive m-imitations are allowed by one and the same player.

Example 3 Suppose that the game is $(1, 2)$-Imitation Nim, so that no 2-imitation is allowed. Then if the starting position is $(1, 2)$ and Alice moves to $(0, 2)$, Bob cannot move, hence $(1, 2)$ is an N-position and it must be non-dynamic since $(1, 2) \to (0, 2)$ is always an option regardless of whether there was a previous move or not.

1.4.4. *The dual of (1,m)-Imitation Nim.* Fix a positive integer m. There is a generalisation of Wythoff Nim, see [Fra], here denoted by $(1, m)$-Wythoff Nim, which (as we will show in Section 2) has a natural P-position correspondence with $(1, m)$-Imitation Nim. The rules for this game are: remove any number of tokens from precisely one of the piles, or remove tokens from both piles, say x and y tokens respectively, with the restriction that $|\, x - y \,| < m$.

And indeed, to continue Example 3, $(1, 2)$ is certainly an N-position of $(1, 2)$-Wythoff Nim, since here $(1, 2) \to (0, 0)$ is an option. On the other hand $(1, 3)$ is P—and non-dynamic P of $(1, 2)$-Imitation Nim since if Alice moves $(1, 3) \to (0, 3)$ or $(1, 0)$ it does not prevent Bob from winning and $(1, 3) \to (1, 2)$ or $(1, 1)$ are losers, since Bob may take advantage of the imitation-rule.

In [Fra], the author shows that the P-positions of $(1, m)$-Wythoff Nim are so-called "Beatty pairs" (view for example the appendix, the original papers in [Ray, Bea] or in [Fra] (page 355) of the form $(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$, where $\beta = \alpha + m$, n is a non-negative integer and

$$\alpha = \frac{2 - m + \sqrt{m^2 + 4}}{2}. \tag{4}$$

1.4.5. *The P-positions of (p, m)-Wythoff Nim.* In the game of (p, m)-Wythoff Nim, originally defined in [HeLa] (as p-blocking m-Wythoff Nim), a player may move as in $(1, m)$-Wythoff Nim and block positions as in $(p, 1)$-Wythoff Nim. From this point onwards whenever we write Wythoff's game or $W = W_{p,m}$ we intend (p, m)-Wythoff Nim.

The P-positions of this game can easily be calculated by a minimal exclusive algorithm (but with exponential complexity in succinct input size) as follows: Let X be a set of non-negative integers. Define $\text{mex}(X)$ as the least non-negative integer not in X, formally $\text{mex}(X) := \min\{x \mid x \in \mathbb{N}_0 \setminus X\}$.

Definition 3 Given positive integers p and m, the integer sequences (a_n) and (b_n) are:

$$\begin{aligned} a_n &= \text{mex}\{a_i, b_i \mid 0 \le i < n\}; \\ b_n &= a_n + \delta(n), \end{aligned}$$

where $\delta(n) = \delta_{p,m}(n) := \left\lfloor \frac{n}{p} \right\rfloor m$.

The next result follows almost immediately from this definition. See also [HeLa] (Proposition 3.1 and Remark 1) for further extensions.

Proposition 1 Let $p, m \in \mathbb{N}$.

(a) The P-positions of (p, m)-Wythoff Nim are the pairs (a_i, b_i) and (b_i, a_i), $i \in \mathbb{N}_0$, as in Definition 3;
(b) The sequences $(a_i)_{i\ge 0}^{\infty}$ and $(b_i)_{i\ge p}^{\infty}$ partition $\mathbb{N}_0$ and for $i \in [0, p - 1]$, $a_i = b_i = i$;
(c) Suppose (a, b) and (c, d) are two distinct P-positions of (p, m)-Wythoff Nim with $a \le b$ and $c \le d$. Then $a < c$ implies $b - a \le d - c$ (and $b < d$);
(d) For each $\delta \in \mathbb{N}$, if $m \,|\, \delta$ then $\#\{i \in \mathbb{N}_0 \mid b_i - a_i = \delta\} = p$, otherwise $\#\{i \in \mathbb{N}_0 \mid b_i - a_i = \delta\} = 0$.

The (p,m)-Wythoff pairs from Proposition 1 may be expressed via Beatty pairs if and only if $p \mid m$. In that case one can prove via an inductive argument that the P-positions of (p,m)-Wythoff Nim are of the form

$$(pa_n,\, pb_n),\ (pa_n+1,\, pb_n+1),\ \ldots,\ (pa_n+p-1,\, pb_n+p-1),$$

where (a_n, b_n) are the P-positions for the game $(1, m/p)$-Wythoff Nim (we believe that this fact has not been recognized elsewhere, at least not in [HeLa] or [Had] in its present form).

For any other p and m we did not have a polynomial time algorithm for telling whether a given position is N or P, until recently—while reviewing this article there has been progress on this matter, so there is a polynomial time algorithm, see [Had]. See also a conjecture in [HeLa], Section 5, saying in a specific sense that the (p,m)-Wythoff pairs are "close to" the Beatty pairs $(n\alpha, n\beta)$ where $\beta = \alpha + \frac{m}{p}$ and

$$\alpha = \frac{2p - m + \sqrt{m^2 + 4p^2}}{2p},$$

which is settled for the case $m = 1$ in the appendix. In the general case, as is shown in [Had], the explicit bounds for a_n and b_n are

$$(n - p + 1)\alpha \le a_n \le n\alpha$$

and

$$(n - p + 1)\beta \le b_n \le n\beta.$$

A reader who, at this point, feels ready to plough into the main idea of our result, may move on directly to Section 2—where we state how the winning positions of (p,m)-Imitation Nim correlate to those of (p,m)-Wythoff Nim and give a proof for the case $m = 1$. In Section 3 we finish off with a couple of suggestions for future work.

1.4.6. *Further Examples.* In this section we give two examples of games where $p > 1$ and $m > 1$ (simultaneously), namely in Example 4 $(3,2)$-Imitation Nim and in Example 5 $(3,3)$-Imitation Nim. The style is informal.

In Example 4 the winning strategy (via the imitation rule) is in a direct analogy to the case $m = 1$. In Example 5 we indicate how our relaxation of the imitation rule changes how a player may take advantage of the imitation rule—in a way that is impossible for the case $m = 1$. We illustrate why this does not affect the nice correlation between the winning positions of Imitation Nim and Wythoff's game. Hence these examples may well be studied in connection with (a second reading of) the proof of Theorem 1.

Example 4 The first few P-positions of $(3,2)$-Wythoff Nim are

$$(0,0), (1,1), (2,2), (3,5), (4,6), (7,9), (8,12), (10,14),$$
$$(11,15), (13,19), (16,22), (17,23), (18,26), (20,28).$$

For the moment assume that the first few non-dynamic P-positions of $(3,2)$-Imitation Nim are $(0,0), (3,5), (8,12), (13,19)$ and $(18,26)$.

Suppose the position is $(20,27)$. We 'suspect' that this is a non-dynamic N-position since irrespective of any previous moves, Alice can move $(20,27) \to (17,27)$. This move clearly resets the counter and Alice can make sure that Bob will not reach the non-dynamic P-position $(13, 19)$, because then he would need to imitate Alice's moves 3 times. Is there any other good move for Bob? Since any other Nim-type move would take him to another N-position (as of Wythoff's game), he must try and rely on the imitation rule. So he needs to remove tokens from the pile with 17 tokens. But, however he does this, Alice will, by inspection, be able to reach a P-position (as of Wythoff's game) *without* imitating Bob. Namely, if Bob moves to $(x, 27)$, then Alice next move will be (x, y), where $y \leq x + 6$ and $10 = 27 - 17 > 6 + 3 = 9$, so the move is not an imitation.

By this example we see that the imitation rule is an eminent tool for Alice, whereas Bob is the player who 'suffers its consequences'. In the next example Bob tries to get around his predicament by hoping that Alice would 'rely too strongly' on the imitation rule.

Example 5 The first few P-positions of $(3, 3)$-Wythoff Nim are

$$(0, 0), (1, 1), (2, 2), (3, 6), (4, 7), (5, 8), (6, 12).$$

Suppose, in a game of $(3, 3)$-Imitation Nim, the players have moved

$$\begin{aligned} \text{Alice} &: (6, 9) \to (5, 9) \\ \text{Bob} &: (5, 9) \to (5, 6) \text{ an imitation} \\ \text{Alice} &: (5, 6) \to (4, 6) \\ \text{Bob} &: (4, 6) \to (3, 6) \text{ no imitation.} \end{aligned}$$

Bob will win, in spite of Alice trying to use the imitation rule for her advantage. The mistake is Alice's second move, where she should change her 'original plan' and not continue to try and rely on the imitation rule. For the next variation Bob tries to 'confuse' Alice's strategy by 'swapping piles',

$$\begin{aligned} \text{Alice} &: (3, 3) \to (2, 3) \\ \text{Bob} &: (2, 3) \to (2, 1). \end{aligned}$$

Bob has imitated Alice's move once. If Alice continues her previous strategy by removing tokens from the shorter pile, say by moving $(2, 1) \to (2, 0)$, Bob will imitate Alice's move a second time and win. Now Alice's correct strategy is rather to remove token(s) from the higher pile,

$$\begin{aligned} \text{Alice} &: (2, 1) \to (1, 1) \\ \text{Bob} &: (1, 1) \to (0, 1) \\ \text{Alice} &: (0, 1) \to (0, 0). \end{aligned}$$

Then, Alice has become the player who imitates, but nevertheless wins.

2. The winning strategy of Imitation Nim

For the statement of our main theorem we use some more terminology.

Definition 4 Suppose the constants p and m are given as in Imitation Nim or in Wythoff's game. Then, if $a, b \in \mathbb{N}_0$,

$$\xi(a, b) = \xi_{p,m}\big((a, b)\big) := \#\big\{\ (i, j) \in \mathcal{P}_{W_{p,m}} \ \big|\ j - i = b - a,\ i < a\ \big\}.$$

Then according to Proposition 1(d),

$$0 \le \xi(a,b) \le p,$$

and indeed, if $(a,b) \in \mathcal{P}_{W_{p,m}}$ then $\xi(a,b) < p$ equals the number of P-positions the previous player has to block off (given that we are playing Wythoff's game) in order to win.

Definition 5 Let (a,b) be a position of a game of (p,m)-Imitation Nim. Put

$$L(a,b) = L_{p,m}((a,b)) := p - 1$$

if

(A) (a,b) is the starting position, or
(B) $(c,d) \to (a,b)$ was the most recent move and (c,d) was the starting position, or
(C) The previous move was $(e,f) \to (c,d)$ but the move (or option) $(c,d) \to (a,b)$ is not an m-imitation.

Otherwise, with notation as in (C), put

$$L(a,b) = L(e,f) - 1.$$

Notice that by the definition of (p,m)-Imitation Nim,

$$-1 \le L(a,b) < p,$$

namely it will be convenient to allow $L(a,b) = -1$, although a player cannot move $(c,d) \to (a,b)$ if it is an imitation and $L(e,f) = 0$.

Indeed $L(e,f)$ represents the number of imitations the player moving from (c,d) still has 'in credit'.

Theorem 1 Let $0 \le a \le b$ be integers and suppose the game is (p,m)-Imitation Nim. Then (a,b) is P if and only if

(I) $(a,b) \in \mathcal{P}_{W_{p,m}}$ and $0 \le \xi(a,b) \le L(a,b)$, or
(II) there is a $a \le c < b$ such that $(a,c) \in \mathcal{P}_{W_{p,m}}$ but $-1 \le L(a,c) < \xi(a,c) \le p-1$.

Corollary 1 If (a,b) is a starting position of (p,m)-Imitation Nim it is P if and only if it is a P-position of (p,m)-Wythoff Nim.

Proof of Corollary 1 Put $L(\cdot) = p - 1$ in Theorem 1. □

By Theorem 1(I) and the remark after Definition 5 we get that (a,b) is non-dynamic P if and only if $(a,b) \in \mathcal{P}_W$ and $\xi(a,b) = 0$. On the other hand, if $(a,b) \in \mathcal{N}_W$ it is dynamic if and only if there is a $a \le c < b$ such that $(a,c) \in \mathcal{P}_W$ [2].

[2] While reviewing this article we have found out that, under the assumption that through the course of the game at least one player has always used a perfect strategy, several dynamic positions (only N-positions of Wythoff's game though) 'are non-dynamic N'. In this sense one might want to define the set of *perfect dynamic* positions, the subset

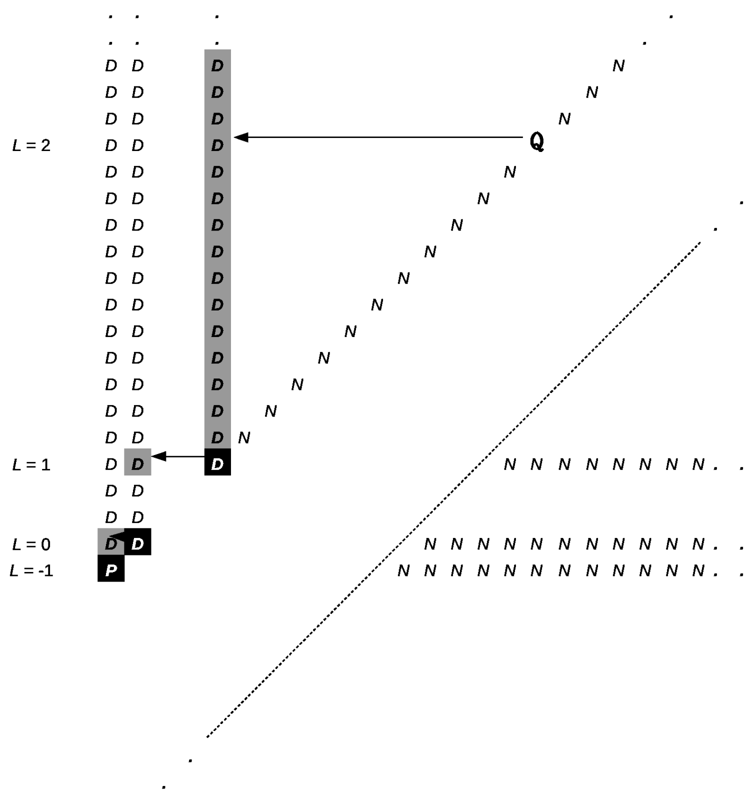

FIGURE 2. A strategy of a game of $(3, 1)$-Imitation Nim. The P is a non-dynamic P-position north of the main diagonal. The black positions are all P-positions of $(3, 1)$-Wythoff Nim on one and the same SW-NE diagonal. The D:s are dynamic positions. The arrows symbolise 3 consecutive winning moves from a position **Q**. A position is grey or black if and only if it is P in some winning strategy (see also footnote 2). The N:s are non-dynamic N-positions.

Proof of Theorem 1 We only give the proof for the case $m = 1$. In this way we may put a stronger emphasis on the idea of the game, at the expense of technical details. Whenever we refer to Proposition 1(b, c or d) we also intend Proposition 1(a).

Suppose (a, b) is as in (I). Then we need to show that, if (x, y) is an option of (a, b) then (x, y) is neither of form (I) nor (II).

But Proposition 1(b) gives immediately that $(x, y) \in \mathcal{N}_W$ so suppose (x, y) is of form (II). Then there is a $x \leq c < y$ such that $(x, c) \in \mathcal{P}_W$ and $L(x, c) < \xi(x, c)$. Since, by (I), $\xi(a, b) \leq L(a, b)$ and $L(a, b) - 1 \leq L(x, c)(\leq L(a, b))$ we get that

$$\xi(a, b) \leq L(a, b) \leq L(x, c) + 1 \leq \xi(x, c),$$

of dynamic positions that are P in some (perfect) strategy, see also Section 3 and Figure 2.

which, in case $c - x = b - a$, is possible if and only if $\xi(a, b) = \xi(x, c)$. But then, since, by our assumptions, $(x, c) \in \mathcal{P}_W$ and $(a, b) \in \mathcal{P}_W$, we get $(a, b) = (x, c)$, which is impossible.

So suppose that $c - x \neq b - a$. Then, by Proposition 1(c), $c - x < b - a$. We have 2 possibilities:

$y = b$: Then if $(x, b) \to (x, c)$ is an imitation of $(a, b) \to (x, b)$ we get $b-c > a - x = b - c$, a contradiction.

$x = a$: For this case the move $(a, y) \to (a, c)$ cannot be an imitation of $(a, b) \to (a, y)$ since the previous player removed tokens from the larger pile. Then $L(a, c) = p-1 \geq \xi(a, c)$ since, by (II), $(a, c) \in \mathcal{P}_W$.

Hence we may conclude that if (a, b) is of form (I) then an option of (a, b) is neither of form (I) nor (II).

Suppose now that (a, b) is of form (II). Then $(a, c) \in \mathcal{P}_W$ is an option of (a, b) but we have $L(a, c) < \xi(a, c)$ so (a, c) is not of form (I). Since $(a, c) \in \mathcal{P}_W$, by Proposition 1(b), it cannot be of form (II). But then, since $b > c$, by Proposition 1(b) and (c), any other option of (a, b), say (x, y) must be an N-position of Wythoff's game so suppose (x, y) is of form (II). We get two cases:

$y = b$: Then $0 \leq x < a$ and there is an option $(x, d) \in \mathcal{P}_W$ of (x, b) with $x \leq d < b$, but by Proposition 1(b) and (c) $d - x \leq c - a < b - a$ so that $(x, b) \to (x, d)$ does not imitate $(a, b) \to (x, b)$. Hence $L(x, d) = p - 1 \geq \xi(x, d)$, which contradicts the assumptions in (II).

$x = a$: Then $0 \leq y < b$. But then, if $y > c$, $(a, c) \in \mathcal{P}_W$ is an option of (a, y) and two consecutive moves from the larger pile would give $L(a, c) = p - 1 \geq \xi(a, c)$. Otherwise, by Proposition 1(b), there is no option of (a, y) in $\mathcal{P}_W$. In either case a contradiction to the assumptions in (II).

We are done with the first part of the proof.

Therefore, for the remainder of the proof, assume that (α, β), $0 \leq \alpha \leq \beta$, is neither of form (I) nor (II). Then

(i) $(\alpha, \beta) \in \mathcal{P}_W$ implies $0 \leq L(\alpha, \beta) < \xi(\alpha, \beta) \leq p - 1$, and
(ii) there is a $\alpha \leq c < \beta$ such that $(\alpha, c) \in \mathcal{P}_W$ implies $0 \leq \xi(\alpha, c) \leq L(\alpha, c) \leq p - 1$.

We need to find an option of (α, β), say (x, y), of form (I) or (II).

If $(\alpha, \beta) \in \mathcal{P}_W$, by Proposition 1(b), (ii) is trivially satisfied, and by (i) $\xi(\alpha, \beta) > 0$, so there is a position $(x, z) \in \mathcal{P}_W$ such that $z - x = \beta - \alpha$ with $x \leq z < \beta (= y)$. Then, since $L(\alpha, \beta) < \xi(\alpha, \beta)$, the option (x, β) satisfies (II) (and hence, by the imitation rule, $(\alpha, \beta) \to (x, \beta)$ is the desired winning move).

For the case $(\alpha, \beta) \in \mathcal{N}_W$ (here (i) is trivially true), suppose $(\alpha, c) \in \mathcal{P}_W$ with $\alpha \leq c < \beta$. Then (ii) gives $L(\alpha, c) \geq \xi(\alpha, c)$, which clearly holds for example if the most recent move was no imitation. In any case it imediately implies (I).

If $c < \alpha$, with $(\alpha, c) \in \mathcal{P}_W$, then (ii) holds trivially by Proposition 1(b) and so (I) holds because $(\alpha, \beta) \to (\alpha, c)$ is no imitation (since if it was, the previous move must have been from the larger pile).

If $c < \alpha$ with $(c, \beta) \in \mathcal{P}_W$ the move $(\alpha, \beta) \to (c, \beta)$ is no imitation since tokens have been removed from the smaller pile. Hence $p - 1 = L(c, \beta) \geq \xi(c, \beta)$.

The only remaining case for (α, β) an N-position of Wythoff's game is whenever there is a position $(x, z) \in \mathcal{P}_W$ such that $x < \alpha$ and

$$\beta - \alpha = z - x. \tag{5}$$

We may assume there is no $c < \beta$ such that $(\alpha, c) \in \mathcal{P}_W$ (since we are done with this case). Then (ii) holds trivially and by Proposition 1(b) there must be a $c > \beta$ such that $(\alpha, c) \in \mathcal{P}_W$. But then, by Proposition 1(c) and (d), we get $\xi(\alpha, \beta) = p > 0$ and so, since we for this case may take (x, z) such that $p - 1 = \xi(x, z)$, we get $L(x, z) \leq p - 2 < \xi(x, z)$. Then, by (5), clearly $(x, \beta) = (x, y)$ is the desired position of form (II). □

3. Final questions

Let us finish off with some questions.

- Consider a slightly different setting of an impartial game, namely where the second player does not have perfect information, but the first player (who has) is not aware of this fact—similar settings have been discussed in for example [BeCoGu, Ow]. We may ask, for which games (start with the games we have discussed) is there a simple *second player's strategy* which lets him *learn* the winning strategy of the game *while playing*—in the sense that if he starts a new 'partie' of the same game at least 'one move after' the first one, he will win?
- Is there a generalisation of Wythoff Nim to $n > 2$ piles of tokens (see for example [BlFr, FrKr, Su, SuZe]), together with a generalisation of 2-pile Imitation Nim, such that the P-positions correlate (at least as starting positions)?
- Are there other impartial (or partizan) games where an imitation rule corresponds in a natural way to a blocking manoeuvre?
- Can one formulate a general rule as to when such correspondences can be found and when not?

Acknowledgements At the Integers 2007 Conference when I introduced Imitation Nim to Aviezri Fraenkel I believe he quickly responded "*Limitation Nim*". As this name emphasises another important aspect of the game, I would like to propose it for the "dual" of Wythoff's original game, that is whenever one wants to emphasise that no imitation is allowed. I would also like to thank A. Fraenkel for contributing with some references.

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Appendix

Peter Hegarty

The purpose of this appendix is to provide a proof of Conjecture 5.1 of [HeLa] in the case $m = 1$, which is the most natural case to consider. Notation concerning 'multisets' and 'greedy permutations' is consistent with Section 2 of [HeLa]. We begin by recalling

Definition : Let r, s be positive irrational numbers with $r < s$. Then (r, s) is said to be a *Beatty pair* if

$$\frac{1}{r} + \frac{1}{s} = 1. \tag{6}$$

Theorem *Let (r, s) be a Beatty pair. Then the map $\tau : \mathbb{N} \to \mathbb{N}$ given by*

$$\tau([nr]) = [ns], \ \ \forall\, n \in \mathbb{N}, \qquad \tau = \tau^{-1},$$

is a well-defined involution of $\mathbb{N}$. If M is the multiset of differences $\pm\{[ns] - [nr] : n \in \mathbb{N}\}$, then $\tau = \pi_g^M$. M has asymptotic density equal to $(s - r)^{-1}$.

Proof : That τ is a well-defined permutation of $\mathbb{N}$ is Beatty's theorem. The second and third assertions are then obvious.

Proposition *Let $r < s$ be positive real numbers satisfying (7), and let $d := (s - r)^{-1}$. Then the following are equivalent*
(i) r is rational
(ii) s is rational
(iii) d is rational of the form $\frac{mn}{m^2 - n^2}$ for some positive rational m, n with $m > n$.

Proof : Straightforward algebra exercise.

Notation : Let (r, s) be a Beatty pair, $d := (s - r)^{-1}$. We denote by M_d the multisubset of $\mathbb{N}$ consisting of all differences $[ns] - [nr]$, for $n \in \mathbb{N}$. We denote $\tau_d := \pi_g^{\pm M_d}$.

As usual, for any positive integers m and p, we denote by $\mathcal{M}_{m,p}$ the multisubset of $\mathbb{Z}$ consisting of p copies of each multiple of m and $\pi_{m,p} := \pi_g^{\mathcal{M}_{m,p}}$. We now denote by $M_{m,p}$ the submultiset consisting of all the positive integers in $\mathcal{M}_{m,p}$ and $\overline{\pi}_{m,p} := \pi_g^{\pm M_{m,p}}$. Thus

$$\overline{\pi}_{m,p}(n) + p = \pi_{m,p}(n + p) \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

Since $\mathcal{M}_{m,p}$ has density p/m, there is obviously a close relation between $M_{m,p}$ and $M_{p/m}$, and thus between the permutations $\pi_{m,p}$ and $\tau_{p/m}$. The precise nature of this relationship is, however, a lot less obvious on the level of permutations. It is the purpose of the present note to explore this matter.

We henceforth assume that $m = 1$.

To simplify notation we fix a value of p. We set $\pi := \overline{\pi}_{1,p}$, Note that

$$r = r_p = \frac{(2p-1)+\sqrt{4p^2+1}}{2p}, \quad s = s_p = r_p + \frac{1}{p} = \frac{(2p+1)+\sqrt{4p^2+1}}{2p}.$$

FURTHER NOTATION : If X is an infinite multisubset of $\mathbb{N}$ we write $X = (x_k)$ to denote the elements of X listed in increasing order, thus strictly increasing order when X is an ordinary subset of $\mathbb{N}$. The following four subsets of $\mathbb{N}$ will be of special interest :

$$\begin{aligned} A_\pi &:= \{n : \pi(n) > n\} := (a_k), \\ B_\pi &:= \mathbb{N}\backslash A_\pi := (b_k), \\ A_\tau &:= \{n : \tau(n) > n\} := (a_k^*), \\ B_\tau &:= \mathbb{N}\backslash A_\tau := (b_k^*). \end{aligned}$$

Note that $b_k = \pi(a_k)$, $b_k^* = \tau(a_k^*)$ for all k. We set

$$\epsilon_k := (b_k - a_k) - (b_k^* - a_k^*) = (b_k - b_k^*) - (a_k - a_k^*).$$

Lemma 1 *(i) For every $n > 0$,*

$$|M_p \cap [1,n]| = |M_{1,p} \cap [1,n]| + \epsilon,$$

where $\epsilon \in \{0, 1, ..., p-1\}$.
(ii) $\epsilon_k \in \{0,1\}$ for all k and if $\epsilon_k = 1$ then $k \not\equiv 0 \ (mod\ p)$.
(iii) $a_{k+1}^* - a_k^* \in \{1,2\}$ for all $k > 0$ and cannot equal one for any two consecutive values of k.
(iv) $b_{k+1}^* - b_k^* \in \{2,3\}$ for all $k > 0$.

PROOF : (i) and (ii) are easy consequences of the various definitions. (iii) follows from the fact that $r_p \in (3/2, 2)$ and (iv) from the fact that $s_p \in (2,3)$.

Main Theorem *For all $k > 0$, $|a_k - a_k^*| \le p - 1$.*

REMARK : We suspect, but have not yet been able to prove, that $p-1$ is best-possible in this theorem.

PROOF OF THEOREM : The proof is an induction on k, which is most easily phrased as an argument by contradiction. Note that $a_1 = a_1^* = 1$. Suppose the theorem is false and consider the smallest k for which $|a_k^* - a_k| \ge p$. Thus $k > 1$.

Case I : $a_k - a_k^* \ge p$.

Let $a_k - a_k^* := p' \ge p$. Let b_l be the largest element of B_π in $[1, a_k)$. Then $b_{l-p'+1}^* > a_k^*$ and Lemma 1(iv) implies that $b_l^* - b_l \ge p'$. But Lemma 1(ii) then implies that also $a_l^* - a_l \ge p' \ge p$. Since obviously $l < k$, this contradicts the minimality of k.

Case II : $a_k^* - a_k \ge p$.

Let $a_k^* - a_k := p' \ge p$. Let b_l^* be the largest element of B_τ in $[1, a_k^*)$.

Then $b_{l-p'+1} > a_k$. Lemma 1(iv) implies that $b_{l-p'+1} - b^*_{l-p'+1} \geq p'$ and then Lemma 1(ii) implies that $a_{l-p'+1} - a^*_{l-p'+1} \geq p' - 1$. The only way we can avoid a contradiction already to the minimality of k is if all of the following hold :

(a) $p' = p$.
(b) $b^*_i - b^*_{i-1} = 2$ for $i = l, l-1, ..., l-p+2$.
(c) $l \not\equiv -1 \pmod p$ and $\epsilon_{l-p+1} = 1$.

To simplify notation a little, set $j := l-p+1$. Now $\epsilon_j = 1$ but parts (i) and (ii) of Lemma 1 imply that we must have $\epsilon_{j+t} = 0$ for some $t \in \{1, ..., p-1\}$. Choose the smallest t for which $\epsilon_{j+t} = 0$. Thus

$$b^*_j - a^*_j = b^*_{j+1} - a^*_{j+1} = \cdots = b^*_{j+t-1} - a^*_{j+t-1} = (b^*_{j+t} - a^*_{j+t}) - 1.$$

From (b) it follows that

$$a^*_{j+t} - a^*_{j+t-1} = 1, \ \ a^*_{j+\xi} - a^*_{j+\xi-1} = 2, \ \xi = 1, ..., t-1. \tag{8}$$

Let b^*_r be the largest element of B_τ in $[1, a^*_j)$. Then from (9) it follows that

$$b^*_{r+t} - b^*_{r+t-1} = 3, \ \ b^*_{r+\xi} - b^*_{r+\xi-1} = 2, \ \xi = 2, ..., t-1. \tag{9}$$

Together with Lemma 1(iv) this implies that

$$b^*_{r+p-1} - b^*_{r+1} \geq 2p - 3. \tag{10}$$

But since $a^*_j = a_j - (p-1)$ we have that $b_{r+p-1} < a_j$. Together with (11) this enforces $b^*_{r+p-1} - b_{r+p-1} \geq p$, and then by Lemma 1(ii) we also have $a^*_{r+p-1} - a_{r+p-1} \geq p$. Since it is easily checked that $r+p-1 < k$, we again have a contradiction to the minimality of k, and the proof of the theorem is complete.

This theorem implies Conjecture 5.1 of [HeLa]. Recall that the P-positions of $(p,1)$-Wythoff Nim are the pairs $(n-1, \pi_{1,p}(n) - 1)$ for $n \geq 1$.

Corollary *With*

$$L = L_p = \frac{s_p}{r_p} = \frac{1 + \sqrt{4p^2+1}}{2p}, \quad l = l_p = \frac{1}{L_p},$$

we have that, for every $n \geq 1$,

$$\pi_{1,p}(n) \in \{\lfloor nL \rfloor + \epsilon, \lfloor nl \rfloor + \epsilon : \epsilon \in \{-1, 0, 1, 2\}\}. \tag{11}$$

PROOF : We have $\pi_{1,p}(n) = n$ for $n = 1, ..., p$, and one checks that (12) thus holds for these n. For $n > p$ we have by (8) that

$$\pi_{1,p}(n) = \pi(n-p) + p, \tag{12}$$

where $\pi = \overline{\pi}_{1,p}$. There are two cases to consider, according as to whether $n - p \in A_\pi$ or B_π. We will show in the former case that $\pi_{1,p}(n) = \lfloor nL \rfloor + \epsilon$ for some $\epsilon \in \{-1, 0, 1, 2\}$. The proof in the latter case is similar and will be omitted.

So suppose $n - p \in A_\pi$, say $n - p = a_k$. Then

$$\pi(a_k) = b_k = a_k + (b^*_k - a^*_k) + \epsilon_k. \tag{13}$$

Moreover $a_k^* = \lfloor kr_p \rfloor$ and $b_k^* = \lfloor ks_p \rfloor$, from which it is easy to check that

$$b_k^* = a_k^* L + \delta, \quad \text{where } \delta \in (-1, 1).$$

Substituting into (14) and rewriting slightly, we find that

$$\pi(a_k) = a_k L + (a_k^* - a_k)(L - 1) + \delta + \epsilon_k,$$

and hence by (13) that $\pi_{1,p}(n) = nL + \gamma$ where

$$\gamma = (a_k^* - a_k - p)(L - 1) + \delta + \epsilon_k.$$

By Lemma 1, $\epsilon_k \in \{0, 1\}$. By the Main Theorem, $|a_k^* - a_k| \leq p - 1$. It is easy to check that $(2p - 1)(L - 1) < 1$. Hence $\gamma \in (-2, 2)$, from which it follows immediately that $\pi_{1,p}(n) - \lfloor nL \rfloor \in \{-1, 0, 1, 2\}$. This completes the proof.

Remark : As stated in Section 5 of [HeLa], computer calculations seem to suggest that, in fact, (12) holds with just $\epsilon \in \{0, 1\}$. So once again, the results presented here may be possible to improve upon.

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