**Multivectors, Spinors and Index Theorems, 10 hp**

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| Course period: | Last day for application: |
| Periods 1 and 2, 2014, starting September 1st | Not applicable |
| Course leader / Address for applications: | |
| Andreas Rosén, andreas.rosen@chalmers.se | |
| Course description (Advertisment for Ph.D. students): | |
| The main goal of the course will be to work through a very down-to-earth proof of the index theorems for the Hodge-Dirac (de Rham) operator, and the Atiyah-Singer (spinor) Dirac operator on compact manifolds. We will take this as an excuse to properly study some fundamental algebraic and geometric constructions, for which a strange dichotomy exists. On the one hand, they have never entered the basic mathematics curriculum properly despite being 100-200 years old by now. For this reason they are surprisingly unknown to most students of mathematics. On the other hand, these constructions of Grassmann, Clifford and Cartan which we refer to, are used in a rather abstract way in some fields of advanced mathematical research. We will try to bridge the gap between these two extremes.  The first and basic construction is the exterior algebra of multivectors: The higher dimensional bivectors and k-vectors beyond the boring one dimensional vectors that everyone knows. We are not going to follow Cartan in focusing algebraically on the dual alternating forms, but build our geometric intuition about multivectors properly. A second construction is the Clifford product of multivectors in an inner product space, for example euclidean space or Minkowski spacetime. This constitutes the real non-commutative higher dimensional analogue of the complex algebra in the euclidean plane. However, algebraically the Clifford product behaves like a matrix product. This leads us to a third basic construction, namely that of spinors, the square roots of multivectors, which are needed for the Atiyah-Singer Dirac operator in particular.  After our study of the algebra and geometry of exterior algebra, Clifford algebra and spinors, we proceed to multivector analysis. For the index theorems, we will need some basic differential and integral calculus for multivectors on manifolds. The Atiyah-Singer index theorem is usually regarded as one of the main achievement of 20th century mathematics, and it is easy to get lost in seemingly abstract mathematics when studying this. As indicated above, we will we have a very concrete goal in this course: For two different Dirac operators we want to prove a formula for the index of this Fredholm operator, which involves an integral over the manifold with an integrand calculated locally from the curvature tensor. My main source of inspiration is the beautiful book ``The index theorem and the heat equation method'' by Yanlin Yu. Hopefully the course will inspire and well equip you for further studies in this area. Remember: The human being is an expert on generalisation and abstraction, it is the basic ideas and the concrete examples which are hard to get! | |

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| Responsible department and other participation departments/organisations: |
| Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg |
| Teacher: |
| Andreas Rosén |
| Andreas Rosén |

Faculty of Science; Department of Mathematical Sciences

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*Third cycle education*

**1. Confirmation**

The syllabus was confirmed by the Head of the Department

Disciplinary domain: Science

Department in charge: Department of Mathematical Sciences

Main field of study: Mathematics

**2. Position in the educational system**

Elective course; third-cycle education.

**3. Entry requirements**

Admitted to third cycle education. The course is also possible for interested master students with appropriate mathematical prerequisites.

**4. Course content**

Exterior algebra and Clifford algebra of multivectors, spinors and representations of Clifford algebra, exterior differentiation and integration, Hodge decompositions of multivector fields, Dirac operators on vector bundles over compact manifolds, Chern-Gauss-Bonnet’s index theorem and the index theorem for the Atiyah-Singer spin-Dirac operator.

**5. Outcomes**

After completion of the course the Ph.D. student is expected to ...

1. Knowledge and understanding

- have a good understanding of multivectors and spinors, their algebraic constructions and geometric interpretations

- know differential and integral multivector calculus and how it generalizes classical vector calculus and complex analysis

- know the two index theorems for Dirac operators covered by the course

2. Skills and abilities

- be able to perform concrete calculations with multivectors and spinors

- be able to perform concrete calculations in frames for vector bundles over manifolds, using multivector calculus

3. Judgement and approach

- be able to see how to solve many mathematical problems with the powerful multivector language, and not only with the classical vector language

**6. Required reading**

Andreas Rosén: “Geometric multivector analysis”, version 3. (Available at course start.)

Lecture notes

**7. Assessment**

1. An oral exam at the end of the course: the student is expected to account for the theory covered during the course. Towards the end of the course a precise list of the results that the student may be tested on, will be handed out.
2. Weekly home assignments to be handed in written solutions to.

For a pass grade, the student will have to have solved the assignments satisfactorily and passed the oral exam.

**8. Grading scale**

The grading scale comprises Fail, (U), Pass (G)

**9. Course Evaluation**

The course evaluation is carried out together with the Ph.D. students at the end of the course, and is followed by an individual, anonymous survey. The results and possible changes in the course will be shared with the students who participated in the evaluation and to those who are beginning the course.

**10. Language of instruction**

The language of instruction is English.