Soluble groups with an automorphism inverting many elements

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Abstract

Let G be a finite soluble group. We derive upper bounds, in terms of the derived length of G, for the maximal proportion of elements of G which can be sent to their inverses under a group automorphism.

1. Introduction and notation

Let G be a finite group. If α is an automorphism of G, we follow [4] in denoting by $I_G(\alpha)$ the set of those elements of G which are sent to their inverses under α, i.e.: $I_G(\alpha) = \{g \in G \mid g\alpha = g^{-1}\}$. We then set $l(\alpha, G) = \frac{|I_G(\alpha)|}{|G|}$, and define

$$l(G) = \max_{\alpha \in \mathrm{Aut}(G)} l(\alpha, G). \tag{1}$$

There are several papers in the literature which investigate the properties of the function $l(o)$ from finite groups to $\mathbf{Q} \cap (0, 1]$. The idea underlying these investigations is that, if $l(G)$ is 'large', then G should be 'close' to being abelian. The plausibility of this idea stems from the simple observation that if $l(G) = 1$, then G is abelian. The first non-trivial result was due to Manning [6], who proved that if G is not abelian, then $l(G) \leq \frac{3}{4}$. Much later, Liebeck and MacHale ([4], Theorem 4.13) classified all G for which $l(G) > \frac{1}{2}$. These groups have a very restricted structure. For our purposes, their most noteworthy feature is that they are all soluble of length at most 2. More recently, this classification has been extended [2] [3] to include all groups G for which $l(G) = \frac{1}{2}$. The only other significant result seems to be due to Potter [7], who proved that if $l(G) > \frac{4}{15}$, then G is soluble.

Let us now explain the motivation for the results to appear in this paper. First, notation : We shall denote by Ξ_n the class of finite soluble groups whose derived length is n. We then define the real number l_n by

$$l_n = \sup_{G \in \Xi_n} l(G). \tag{2}$$

The results quoted above imply that $l_1 = 1$ (trivially), $l_2 = \frac{3}{4}$ and $l_3 \leq \frac{1}{2}$. In fact, from the result of [2], it is easy to find $G \in \Xi_3$ such that $l(G) = \frac{1}{2}$, and thus to see that $l_3 = \frac{1}{2}$. The smallest such group has the following presentation

$$G_1 = < a, b, x \mid a^2 = b^2 = z \in Z(G), z^2 = x^3 = 1, b^{-1}ab = a^{-1}, x^{-1}ax = b, x^{-1}bx = ba > . \tag{3}$$

G_1 has order 24, and is a semi-direct product of the quaternion group Q by C_3. It is isomorphic to the group $\mathrm{SL}(2,3)$ of 2×2 matrices over the field $\mathbf{F}_3$ with determinant one.

The map $\alpha : G_1 \to G_1$ defined by

$$a\alpha = a^{-1} \quad b\alpha = ab \quad x\alpha = x^{-1}, \tag{4}$$

is easily checked to extend to an automorphism of G_1, and to satisfy $l(\alpha, G_1) = \frac{1}{2}$. In fact, α inverts elementwise the six cosets of $Z(G_1)$ represented by the elements $1, a, x^{\pm 1}, (bx)^{\pm 1}$.

These observations led us to wonder as to the value of l_n for arbitrary n. We must note at this point that we still have no idea what the precise value of l_n is for any $n > 4$. The furthest we have got is to show that $l_4 = \frac{3}{8}$. This fact is proven in Section 2.

But instead of asking for the precise value of l_n, it is quite natural just to ask for a reasonable upper bound. This question seems to be far more tractable, as indicated by the following basic result :

Theorem 1.1. *The sequence l_n tends to zero as $n \to \infty$. In fact, $l_{3+2k} \leq \frac{1}{2}(\frac{3}{4})^k$ for all $k \geq 0$. More generally, $l_n \leq l_p l_q$ whenever $n = p + q$.*

We will now give a very simple proof of this result, which depends on the next lemma. We adopt the following notation : If $I = \mathrm{Inn}(G)$ and $\alpha \in \mathrm{Aut}(G)$, we denote by $l(\alpha I, G)$ the maximum of the fractions $l(\alpha\beta, G)$ for $\beta \in I$. Then we have

Lemma 1.2. *Suppose $\alpha \in Aut(G)$ and N is a normal, α-invariant subgroup of G. Let α^* denote the induced automorphism of G/N. Then*

$$l(\alpha, G) \leq l(\alpha I, N) l(\alpha^*, G/N) \tag{5}$$

and, consequently,

$$l(G) \leq l(N)l(G/N). \tag{6}$$

PROOF OF LEMMA : This is a simple extension of Lemma 2.1(b) of [7]. The number of cosets of N in G which intersect $I_G(\alpha)$ is at most $l(\alpha^*, G/N) \cdot \mid G/N \mid$. Pick a coset Ng of N such that $\mid Ng \cap I_G(\alpha) \mid$ is as large as possible. Assume g is chosen to lie in $I_G(\alpha)$. Then $\{n \in N \mid ng \in I_G(\alpha)\} = \{n \in N \mid n\alpha = g^{-1}n^{-1}g\} = \{n \in N \mid (n)\alpha I_{g^{-1}} = n^{-1}\}$. This set has size $l(\alpha I_{g^{-1}}, N) \mid N \mid \leq l(\alpha I, N) \mid N \mid$ (here $I_{g^{-1}}$ denotes the inner automorphism induced by g^{-1}). Thus, $\mid Ng \cap I_G(\alpha) \mid \leq l(\alpha I, N) \mid N \mid$, and so $\mid I_G(\alpha) \mid \leq l(\alpha I, N) \mid N \mid \times l(\alpha^*, G/N) \mid G/N \mid$, from which we obtain equation (5). Then (6) follows immediately, which proves the lemma.

In the proof of the theorem to follow, $G^{(i)}$ denotes the i^{th} derived subgroup of G, with the convention that $G^{(0)} = G$.

PROOF OF THEOREM 1.1 : Note that the inequality for l_{3+2k} can be obtained from the more general inequality by putting $p = 2$, using induction on k and the facts (already mentioned) that $l_2 = \frac{3}{4}$ and $l_3 = \frac{1}{2}$. The proof of the general inequality is very simple. So suppose $G \in \Xi_n$ and pick p and q such that $p + q = n$. Put $H = G^{(q)}$. Then $H \in \Xi_p$ and $G/H \in \Xi_q$, so $l(H) \leq l_p$ and $l(G/H) \leq l_q$. It follows from equation (6) that $l(G) \leq l_p l_q$, as desired.

In particular, setting $k = 1$ (or $p = 2, q = 3$), the theorem tells us that $l_5 \leq \frac{3}{8}$. Also, it gives no information about l_4 beyond the trivial fact that $l_4 \leq l_3 = \frac{1}{2}$. But, as mentioned above, we shall prove in Section 2 that $l_4 = \frac{3}{8}$. In fact, we will show how to generalize an argument used in proving that $l_4 \leq \frac{3}{8}$ to obtain a general bound for l_n (Theorem 2.6) which improves substantially upon Theorem 1.1. For example, Theroem 2.6 will imply that $l_5 \leq \frac{9}{32}$. However, this result is still by no means best-possible. In Section 3, we illustrate this by proving that $l_5 \leq \frac{4}{15}$ (Theorem 3.1), which allows for a stronger formulation of Potter's result (Corollary 3.2). Since our discussion obviously leaves many questions unanswered, we will close Section 3 with some brief suggestions for future work.

2. The value of l_4

In this section we shall obtain a general bound for l_n which improves upon Theorem 1.1. The proof will require induction on n, and to start the induction, we must deal with the case $n = 4$ explicitly. In fact, we can find the precise value of l_4, namely -

Theorem 2.1. $l_4 = \frac{3}{8}$.

We now prove this result. First, we show that $l_4 \geq \frac{3}{8}$ by exhibiting a group $G \in \Xi_4$ such that $l(G) \geq \frac{3}{8}$. The smallest such group has order 48, and is a non-split extension of the group G_1 of equation (3) by C_2. More precisely, our group has presentation

$$G_2 = < G_1, y \mid y^2 = z, y^{-1}ay = a^{-1}, y^{-1}by = ab, y^{-1}xy = x^{-1} > . \qquad (7)$$

It is easily checked that $l(I_y, G_2) = \frac{3}{8}$. Indeed, $|I_{G_1}(I_y)| = \frac{1}{2}|G_1| = 12$, while I_y also inverts ya and yb. Since $(ya)^2 = z$ and $(yb)^2 = a$, so ya and yb have order 4 and 8 respectively and $< ya > \cap G_1 = < z >$ and $< yb > \cap G_1 = < a >$. Hence, $|I_{G_2}(I_y) \cap G_1 y| = (4-2)+(8-4) = 6$, as required.

So, in order to prove our theorem, we must show that if $G \in \Xi_4$, then $l(G) \leq \frac{3}{8}$. (Note that much of this part of the argument will be generalised in the proof of Theorem 2.6 below). The proof is by contradiction. So suppose there exists $G \in \Xi_4$ satisfying $l(G) > \frac{3}{8}$. Pick such a group of smallest possible order. Henceforth, until the end of the proof, we reserve the letter G for this fixed choice of group. We also fix a choice of $\alpha \in \mathrm{Aut}(G)$ such that $l(\alpha, G) > \frac{3}{8}$, and reserve the letter α for this automorphism. To simplify notation, we reserve the letter H for the group $G^{(2)}$. Notice immediately that since $l(G) \leq l(H)l(G/H)$ by Lemma 1.2, and $l(G/H) \leq \frac{3}{4}$, we must have $l(H) > \frac{1}{2}$. Now let us proceed with a sequence of lemmas :

Lemma 2.2. *Suppose A is a normal, α-invariant subgroup of G such that $G/A \in \Xi_3$. Then A is abelian, $A \subseteq Z(G)$, and $A \subset I_G(\alpha)$.*

PROOF : First, we show A is abelian. If not, then $l(A) \leq \frac{3}{4}$. But since $G/A \in \Xi_3$, $l(G/A) \leq \frac{1}{2}$. Then, by Lemma 1.2, $l(G) \leq l(A)l(G/A) \leq \frac{3}{8}$, a contradiction.

Now let $\mid A \mid = m$ and $\mid G/A \mid = n$. Write $G = \bigsqcup_{1 \leq i \leq n} Ax_i$ where, if $Ax_i \cap I_G(\alpha) \neq \phi$, then x_i is chosen to lie in $I_G(\alpha)$. For each i such that $x_i \in I_G(\alpha)$ we have, as in [4] Lemma 3.1, that

$$I_G(\alpha I_{x_i^{-1}}) = I_G(\alpha)x_i^{-1}. \qquad (8)$$

In particular, since A is abelian, $I_A(\alpha I_{x_i^{-1}})$ is a subgroup of A.

Now let u be the number of those cosets of A in G which consist entirely of elements of $I_G(\alpha)$, and let v be the number of other cosets of A which intersect $I_G(\alpha)$. Then, by the above remarks, we can at least say that

$$mu + \frac{m}{2}v > \frac{3}{8}mn. \tag{9}$$

On the other hand, since $G/A \in \Xi_3$ and $l_3 = \frac{1}{2}$, we know that

$$u + v \leq \frac{n}{2}. \tag{10}$$

Combining these inequalities, one easily deduces that $u > \frac{n}{4}$. Put $B = C_G(A)$. If $Ax_i \subseteq I_G(\alpha)$ and $Ax_j \subseteq I_G(\alpha)$, then (8) implies that $Bx_i = Bx_j$. Hence, $(G : B) < 4$. Thus G/B is abelian and since $G \in \Xi_4$, we must have $B \notin \Xi_1 \cup \Xi_2$. Now suppose $(G : B) = t$. Since $u > \frac{n}{4}$, in particular $u \neq 0$. So pick any x_i such that $Ax_i \subseteq I_G(\alpha)$. Then (8) now implies that $l(\alpha I_{x_i^{-1}}, B) > \frac{t}{4}$. Hence, if $t \geq 2$, we obtain $l(B) > \frac{1}{2}$, which yields the contradiction that $B \in \Xi_1 \cup \Xi_2$. Thus $t = 1$ and $A \subseteq Z(G)$. It follows immediately that $A \subset I_G(\alpha)$, which completes the proof of the lemma.

Corollary 2.3. *Let T be a normal, α-invariant subgroup of G properly contained in H. Then T is abelian and $T \subseteq Z(H)$.*

PROOF : Immediate.

Lemma 2.4. *$l(H) > \frac{1}{2}$. $H/Z(H)$ is an elementary abelian 2-group. H' is cyclic of order 2, H/H' is elementary abelian and $Z(H)$ is cyclic of order 2 or 4.*

PROOF : Applying Corollary 2.3 to the subgroup H' of H, we see that $H' \subseteq Z(H)$. Thus H is nilpotent, and $H = H_2 \times O$, where H_2 is a 2-group and O has odd order. Suppose O is non-abelian : then $l(O) \leq \frac{1}{3}$, by Theorem 3.2 of [5]. Thus, $l(H) = l(H_2)l(O) \leq \frac{1}{3}$ also, contradicting the fact that $l(H) > \frac{1}{2}$. Thus O is abelian. Now consider the group G/O. Since O is abelian, $G/O \in \Xi_4$ and $l(G) \leq l(G/O)$ by Lemma 1.2. By minimality of G, we conclude that $O = 1$. This proves that H is a 2-group.

Now if H' is not cyclic of order 2, it has a proper subgroup T_1. By Lemma 2.2, T_1 is an abelian, α-invariant normal subgroup of G. But $G/T_1 \in \Xi_4$ and $l(G) \leq l(G/T_1)$ which contradicts minimality of G.

Similarly, let $T_2 = \{h \in H \mid h^2 \in H'\}$. Then T_2 is a G-normal, α-invariant subgroup of H, not contained in $Z(H)$. By Corollary 2.3, it must coincide with H, which proves that H/H' is elementary abelian.

And finally, if $Z(H)$ were not cyclic, it would have a proper subgroup T_3 not containing H' and the group G/T_3 would contradict minimality of G. So $Z(H)$ is cyclic and, since H/H' has exponent 2, $Z(H)$ must have order 2 or 4.

Lemma 2.5 *(i)* $l(G/H) \leq \frac{2}{3}$ *(ii)* $l(H) = \frac{3}{4}$ *or* $\frac{5}{8}$ *and* H *is of type II in Theorem 4.13 of [4].*

PROOF : (i) We know that $l(G) \leq l(H)l(G/H)$ and that $l(H) \leq \frac{3}{4}$, hence $l(G/H) > \frac{1}{2}$. If $l(G/H) > \frac{2}{3}$ then, by Theorem 4.13 of [4], we must have $l(G/H) = \frac{3}{4}$ and $(G/H)' \cong G'/H \cong C_2$. But then G' would be a finite, non-abelian 2-group with a commutator subgroup of index 2, which is well-known to be impossible.

(ii) Since $l(G/H) \leq \frac{2}{3}$, the inequality $l(G) \leq l(H)l(G/H)$ implies that $l(H) > \frac{9}{16}$. Now the fact that H is a 2-group with cyclic commutator subgroup immediately implies that it has the quoted properties, using Theorem 4.13 of [4].

PROOF OF THEOREM 2.1 : Lemma 2.5(ii) suggests that we divide the remainder of the proof into 2 cases.

Case I : $l(H) = \frac{3}{4}$. By [4], $H/Z(H) \cong C_2 \times C_2$. By Lemma 2.4, H is isomorphic to one of D_4, Q or the central product $Q \circ Z \cong D_4 \circ Z$, where D_4 and Q are the dihedral and quaternion groups of order 8 and Z is cyclic of order 4. We have a natural injection $G/C_G(H) \to \mathrm{Aut}(H)$, whose image is contained in the subgroup $C = \mathrm{Cent}(H)$, consisting of those maps which are trivial on $Z(H)$. If $H \cong D_4$ or Q, then $C = \mathrm{Aut}(H) \cong D_4$ or S_4 respectively. If $H \cong Q \circ Z$, then H has one quaternion and three dihedral subgroups, so that $C = C_{\mathrm{Aut}(H)}(Z) \cong \mathrm{Aut}(Q) \cong S_4$. But $G/C_G(H) \in \Xi_3$, so we must have $H \cong Q$ or $Q \circ Z$ and $G/C_G(H) \cong S_4$. But $C_G(H) \subseteq Z(G)$, by Lemma 2.2, so $G/Z(G)$ is isomorphic to a factor group of S_4, hence isomorphic to S_4. From this one can show that either $G' \cong A_4$ or $G' \cong G_1$ (for a proof, see [1]). If $G' \cong A_4$ then $G^{(3)} = 1$, a contradiction. It remains to show that if $G' \cong G_1$, then $l(G) \leq \frac{3}{8}$. Notice that the group G_2 arises here. More precisely, let $Z(H) = < z >$ where $z^2 = 1$. Then G' is generated by elements a, b, x with the same relations as in (3), and G is generated by $G', Z(G)$ and an element y which acts by conjugation on $G/Z(G)$ like an involution in S_4.

Suppose $l(G) > \frac{3}{8}$ and let α be an automorphism for which $l(\alpha, G) > \frac{3}{8}$. Then $l(\alpha^*, G/Z) > \frac{3}{8}$. But $l(S_4) = \frac{5}{12}$ and the only automorphisms of S_4 inverting ten elements are those inner automorphisms induced by conjugation with a transposition. Hence it must be the case that α inverts $Z(G)$ elementwise, $l(\alpha, G) = l(\alpha^*, G/Z(G)) = \frac{5}{12}$ and the generators a, b, x, y can be chosen so that α inverts each of the ten elements $1, a, x^{\pm 1}, (bx)^{\pm 1}, y, ay, by, aby$, which lie in ten different cosets of $Z(G)$. But now, the fact that α inverts each of a, y and ay implies that $[a, y] = 1$. Similarly, the fact that α inverts each of a, by, aby implies that $[a, by] = 1$. But then $[a, b] = 1$, a contradiction.

Case II : $l(H) = \frac{5}{8}$. By [4], $H/Z(H)$ is elementary abelian of order 16. H has many abelian subgroups of index 4; pick any one and call it A. Now consider the argument in the proof of Lemma 2.2. The α-invariance of A was not used to obtain $u > \frac{n}{4}$. Normality of A in G was only used to obtain equation (10). Our group A may not be normal in G, but since $l(G/H) \leq \frac{2}{3}$, we can still replace equation (10) with the weaker inequality

$$u + v \leq \frac{2n}{3}. \tag{11}$$

Equation (9) still holds, so combining (9) and (11), one easily deduces that $u > \frac{n}{12}$. This, in turn, implies that $(G : C_G(A)) < 12$. But $H \subseteq N_G(A)$ so the order of the group $N_G(A)/C_G(A)$ is divisible by 4. Hence,
$(G : C_G(A)) \leq 8$ and $(G : N_G(A)) \leq 2$ and we have the subnormal sequence $C_G(A) \lhd N_G(A) \lhd G$. One sees that this implies that $G^{(2)} = H \subseteq C_G(A)$, a contradiction, since A is not contained inside $Z(H)$.

This eliminates Case II, and completes the proof of Theorem 2.1.

We now proceed immediately to the desired generalisation of this result :

Theorem 2.6. *For any $n \geq 3$, $l_n \leq \frac{1}{2}(\frac{3}{4})^{n-3}$.*

PROOF : We proceed by induction on n. In Section 1, we showed that $l_3 = \frac{1}{2}$, and have just proven that $l_4 = \frac{3}{8}$. So the theorem holds for $n = 3, 4$. Suppose it holds for $3 \leq n < k$ and consider $G \in \Xi_k$, where $k > 4$. As much of the argument from here on is simply a generalisation of that used to prove Theorem 2.1, we simply outline the required steps :

Step 1 : The proof is by contradiction. Suppose there exists $G \in \Xi_k$ such that $l(G) > \frac{1}{2}(\frac{3}{4})^{k-3}$. Pick such a group of smallest possible order and reserve the letter G for it. Also reserve α to denote a fixed choice of automorphism of G such that $l(\alpha, G) = l(G)$. Put $H = G^{(k-2)}$.

Step 2 : Lemma 2.2 generalizes to the following statement :

Suppose A is a normal, α-invariant subgroup of G such that $G/A \in \Xi_{k-1}$. Then A is abelian, $A \subseteq Z(G)$ and $A \subset I_G(\alpha)$.

This statement is proven by following the proof of Lemma 2.2. To show A is abelian, one must invoke the induction hypothesis for $k-1$. Equation (9) must be replaced by

$$mu + \frac{m}{2}v > \frac{1}{2}(\frac{3}{4})^{k-3}mn \tag{12}$$

and, using the induction hypothesis for $k-1$, we can replace (10) by

$$u + v \leq \frac{1}{2}(\frac{3}{4})^{k-4}. \tag{13}$$

From these inequalities, one gets $u > n\left[\frac{1}{4}(\frac{3}{4})^{k-4}\right]$. Putting $B = C_G(A)$, if $(G : B) = t$ then $l(B) > \frac{t}{4}(\frac{3}{4})^{k-4}$. Since $G \in \Xi_k$, the sum of the derived lengths of B and G/B must be at least k. Using the induction hypothesis for $k-1$ again, one checks readily that this is only possible if $t = 1$, which proves the statement.

Step 3 : This step is new, but is suggested by the argument used to finish with *Case II* of Theorem 2.1. The idea is, even if we drop the assumptions, in *Step 2*, that A is either normal in G or α-invariant, we can still say something provided A is contained in H. Indeed, we have

Lemma 2.7. *Let A be any abelian subgroup of H. Then $(G : C_G(A)) < 16(\frac{4}{3})^{k-5}$.*

Proof : Consider the argument used to prove the statement in *Step 2* above. The α-invariance of A was not used to obtain $u > \left[\frac{1}{4}(\frac{3}{4})^{k-4}\right]$. Normality of A in G was used only to obtain equation (13). But since $l(G/H) \leq \frac{1}{2}(\frac{3}{4})^{k-5}$, by the induction hypothesis for $k-2$ (here is an illustration of why we need to have dealt with $k = 4$ to start the induction) we can replace (13) with the weaker inequality

$$u + v \leq \frac{1}{2}(\frac{3}{4})^{k-5}. \tag{14}$$

Combining (12) and (14) gives $u > n\left[\frac{1}{16}(\frac{3}{4})^{k-5}\right]$, from which the conclusion of the lemma follows.

This result will be used at the last step of the proof.

Step 4 : Corollary 2.3 follows verbatim.

Step 5 : Lemma 2.4 also generalizes verbatim, with the same proof.

Step 6 : Part (i) of Lemma 2.5 is true, but obviously useless, in the general setting. Nevertheless, we can obtain part (ii) easily anyway when $k > 4$. To see this, apply the induction hypothesis for $k-2$ again to obtain $l(G/H) \leq \frac{1}{2}(\frac{3}{4})^{k-5}$. Since $l(G) \leq l(H)l(G/H)$, this already implies that $l(H) > \frac{9}{16}$, which is all we need.

Step 7 : Now divide the remainder of the proof into 2 cases, according to the possible values of $l(H)$. If $l(H) = \frac{3}{4}$ then proceed as before. The fact that $G/C_G(H)$ injects into S_4 actually contradicts the assumption that $k > 4$.

If $l(H) = \frac{5}{8}$ then $H/Z(H)$ is elementary abelian of order 16. By Lemma 2.4, H is isomorphic to one of $Q \circ Q$ or $D_4 \circ Q$ or $Q \circ Q \circ Z \cong Q \circ D_4 \circ Z$, where Z is cyclic of order 4. $G/C_G(H)$ injects into a soluble subgroup of $C = \mathrm{Cent}(H)$. In the Appendix we prove that, for each of the three possiblilities for H, every soluble subgroup of C has derived length at most 4. Hence the same is true of $G/C_G(H)$, Since $C_G(H) \subseteq Z(G)$, it follows that $G^{(5)} = 1$, so we are left with the case $k = 5$.

To finish the proof, we use Lemma 2.7. Pick any abelian subgroup of H of index 4 and call it A. Let $X = \mathrm{Core}_G(N_G(A))$ and consider the subnormal sequence

$$X \cap C_G(A) \lhd X \lhd G. \tag{15}$$

Note that $X/X \cap C_G(A)$ is a subgroup of $N_G(A)/C_G(A)$. Lemma 2.7 implies that $(G : C_G(A)) < 16$. But $H \subseteq N_G(A)$, so $(G : C_G(A))$ is a multiple of 4 and therefore $(G : C_G(A)) \leq 12$. Then either $X = G$, or $(N_G(A) : C_G(A)) = 4$ and G/X is a subgroup of S_3. In either case one sees, from (15), that it is impossible to avoid the contradiction that $G^{(3)} = H \subseteq C_G(A)$.

This completes the proof of Theorem 2.6.

3. The value of l_5

If, in Theorem 1.1, we put $n = 6$ and $p = q = 3$, then we see that $l_6 \leq \frac{1}{4}$. The main result of [7] states that if G is a group such that $l(G) > \frac{4}{15}$, then G is soluble. So we see that, not only is G soluble, but of derived length at most 5. Because of Theorem 2.1, the only remaining question is whether or not there is a group G of derived length 5 satisfying $l(G) > \frac{4}{15}$. Theorem 2.6 only gives the bound $l_5 \leq \frac{9}{32}$. As we shall prove, the answer to our question is negative. In particular, this shows that Theorem 2.6 is nowhere near a best-possible result.

Theorem 3.1. $l_5 \leq \frac{4}{15}$.

For completeness, we state the immediate

Corollary 3.2. *If G is a finite group satisfying $l(G) > \frac{4}{15}$, then G is soluble and of derived length at most* 4.

PROOF OF THEOREM 3.1 : The outline of the proof is pretty much the same as that of Theorem 2.1, but in some respects it simplifies and in others it gets more complicated, so we have to be careful.

Again, the proof is by contradiction. So suppose there exists $G \in \Xi_5$ such that $l(G) > \frac{4}{15}$. We pick such a G of smallest possible order and henceforth, until the end of the proof, we reserve the letter G for this group. We also reserve the letter α for a fixed choice of automorphism of G satisfying $l(\alpha, G) > \frac{4}{15}$. Finally, we put $H = G^{(3)}$.

First, we can prove a weaker version of Lemma 2.2 :

Lemma 3.2 *Suppose A is a normal, abelian, α-invariant subgroup of G such that $G/A \in \Xi_4$. Then $(G : C_G(A)) \leq 2$.*

PROOF : Proceed as in the proof of Lemma 2.2. Note we are assuming this time that A is abelian. We must replace equations (9) and (10) by the equations

$$mu + \frac{m}{2}v > \frac{4}{15}mn \tag{16}$$

and

$$u + v \leq \frac{3n}{8} \tag{17}$$

respectively. Combining these inequalities gives $u > \frac{19}{120}n$. Put $B = C_G(A)$. Then $(G : B) < \frac{120}{19} < 7$. Also, if $(G : B) = t$, then $l(B) > \frac{19}{120}t$. Thus, if $t = 6$, B is abelian and the derived length of G/B is at most 2, which gives the contradiction that $H = \{1\}$. Thus $t \leq 5$, which means G/B is abelian and since $G \in \Xi_5$, the derived length of B is at least 4. Therefore, by Theorem 2.1, $l(B) \leq \frac{3}{8}$ so, in particular, $\frac{19}{120}t < \frac{3}{8}$, which forces $t \leq 2$. This is the desired conclusion.

If, in Lemma 3.2, we remove the assumptions that A is normal and α-invariant, we can still say something, as in Lemma 2.7 :

Lemma 3.3. *Let A be any abelian subgroup of H. Then $(G : C_G(A)) < 30$.*

PROOF : Repeat the argument for Lemma 3.2. Equation (16) still holds. Since A may not be normal in G, equation (17) will not hold, but since $l(G/H) \leq \frac{1}{2}$ we can replace it with the weaker inequality

$$u + v \leq \frac{n}{2}. \tag{18}$$

From inequalities (16) and (18) we deduce that $u > \frac{n}{30}$, which, as before, immediately implies that $(G : C_G(A)) < 30$.

Note that the next result is nearly as strong as Lemmas 2.4 and 2.5 - the only difference is we can't say this time whether $Z(H)$ is cyclic. This is what will make the remainder of the proof more complicated than that of Theorem 2.1 :

Lemma 3.4. *$l(H) = \frac{3}{4}$ or $\frac{5}{8}$ or $\frac{9}{16}$ and H is of type II in Theorem 4.13 of [4].*

PROOF : Since $G/H \in \Xi_3$, $l(G/H) \leq \frac{1}{2}$ and since $l(G) \leq l(H)l(G/H)$, we must have $l(H) > \frac{8}{15}$. Thus, in order to prove the lemma it suffices, by Theorem 4.13 of [4], to show that H is a 2-group and H' is cyclic of order 2.

To show H is a 2-group, apply Lemma 3.2 with $A = H'$. Since $(G : C_G(H')) \leq 2$, in particular $H \subseteq C_G(H')$, i.e.: $H' \subseteq Z(H)$. Thus H is nilpotent, and one repeats the argument in the proof of Lemma 2.4 to show H is a 2-group.

So it remains to show H' is cyclic of order 2. It must be elementary abelian, as otherwise $G/\Omega_1(H') \in \Xi_5$ and $l(G) \leq l(G/\Omega_1(H'))$, which contradicts minimality of G. Next, since $I_G(\alpha) \subseteq C_G(\alpha^2)$, $(G : C_G(\alpha^2)) < \frac{15}{4}$, whence $(G : C_G(\alpha^2)) \leq 3$. Therefore, $(G : \mathrm{Core}_G(C_G(\alpha^2))) \leq 6$. From this we infer that $H' \subseteq C_G(\alpha^2)$, so α has order at most 2 on H' (of course, the same is true of H : this will be used later). Now since H' is elementary abelian and $(G : C_G(H')) \leq 2$, H' has an odd number of maximal subgroups which are normal in G. Since α acts as a permutation of these of order at most 2, we conclude that there is a maximal subgroup N of H' which is both normal in G and α-invariant. Then $G/N \in \Xi_5$ and $l(G) \leq l(G/N)$, so by minimality of G we must have $N = \{1\}$. This shows that H' has order 2 and completes the proof of the lemma.

As in Section 2, we now split the remainder of the proof of Theorem 3.1 into three cases, as suggested by Lemma 3.4.

Case I : $l(H) = \frac{3}{4}$. By [4], $H/Z(H) \cong C_2 \times C_2$. Note that $Z(H) \cap Z(G)$ is contained in $I_G(\alpha)$, hence this group must be cyclic, as otherwise any subgroup T of $Z(H) \cap Z(G)$ not containing H' would satisfy $G/T \in \Xi_5$ and $l(G/T) \geq l(G)$, thus contradicting minimality of G. In particular, if $Z(H) \subseteq Z(G)$, then $Z(H)$ is cyclic. Suppose this is the case. We have a natural injection $G/C_G(H) \to \mathrm{Aut}(H)$. But $\mathrm{Aut}(H)$ is soluble of length at most 3, because $\mathrm{Cent}(H)$ is abelian (since $Z(H)$ is cyclic) and $\mathrm{Aut}(H)/\mathrm{Cent}(H)$ is isomorphic to a subgroup of S_3. Thus $G/C_G(H)$ is soluble of length at most 3, yielding the contradiction that $G^{(3)} = H \subseteq C_G(H)$.

We may thus suppose that $(G : C_G(Z(H))) = 2$. Let A be a maximal subgroup of H containing $Z(H)$. If we could choose A to be normal in G then, since α has order at most 2 on H, we could also choose A to be α-invariant, whence A would satisfy the hypotheses of Lemma 3.2. Therefore, we'd have $(G : C_G(A)) \leq 2$, in particular $H \subseteq C_G(A)$, a contradiction. Therefore, A cannot be chosen to be normal in G, so fixing any choice of A, we have $(G : N_G(A)) = 3$. We can still apply Lemma 3.3 to A and obtain $(G : C_G(A)) < 30$. Therefore, $N_G(A)/C_G(A)$ has order less than 10, and since $H \subseteq N_G(A)$, the factor group must have even order, hence order at most 8. Let $X = \mathrm{Core}_G(N_G(A))$. We have a subnormal sequence $X \cap C_G(A) \lhd X \lhd G$. Clearly, $G/X \cong S_3$. If $N_G(A)/C_G(A)$ is abelian, so is $X/X \cap C_G(A)$ and once again we get the contradiction that $G^{(3)} = H \subseteq C_G(A)$. We conclude that $N_G(A)/C_G(A)$ is non-abelian of order 8.

Now, $N_G(A) \cap C_G(Z(H))$ is a subgroup of $N_G(A)$ of index 2. Note that if $g \in N_G(A) \cap C_G(Z(H))$ then $g^2 \in C_G(A)$. This means $N_G(A)/C_G(A)$ cannot be the quaternion group, hence must be D_4. Since D_4 has 5 involutions, it follows that there exists $g \in N_G(A) \backslash C_G(Z(H))$ such that $g^2 \in C_G(A)$.

Next, we claim there exists $a \in A \backslash Z(H)$ such that $a^2 \in H'$. Evidently, it suffices to prove this for any conjugate of A. Choose any $x \in C_G(Z(H)) \backslash N_G(A)$. Pick any $a \in A \backslash Z(H)$ and let $(a)I_x = b$. Since $x \in C_G(Z(H))$, we get $a^2 = b^2$ which implies that $(ab^{-1})^2 \in H'$. Since $ab^{-1} \notin Z(H)$, this establishes our claim.

Now choose any $a \in A \backslash Z(H)$ such that $a^2 \in H'$. Also choose $g \in N_G(A) \backslash C_G(Z(H))$ such that $g^2 \in C_G(A)$. We have $(a)I_g = za$ for some $z \in Z(H)$. Since $g \notin \langle H, C_G(A) \rangle$, $z \notin H'$. But since $a^2 \in H'$, we have

$z^2 = 1$. And since $(a)I_{g^2} = a$, a simple computation shows that $(z)I_g = z$. This implies that $z \in Z(G)$, contradicting the fact that $Z(H) \cap Z(G)$ must be cyclic.

So with this final contradiction, we have eliminated *Case I.*

Case II : $l(H) = \frac{5}{8}$. By [4], $H/Z(H)$ is elementary abelian of order 16, generated by a_1, a_2, x_1, x_2 subject to the relations $[a_1, a_2] = [x_1, x_2] = [a_1, x_2] = [a_2, x_1] = 1$, $[a_1, x_1] = [a_2, x_2] = z$ where $< z > = H'$ is cyclic of order 2. From these relations, one quickly checks that there are precisely 15 abelian subgroups of H of index 4 containing $Z(H)$. Since α acts on these as a permutation of order at most 2, at least one of them is α-invariant. So fix a choice of abelian subgroup A of H of index 4, containing $Z(H)$, such that $A\alpha = A$. The α-invariance of A will be used at one point in the argument to follow.

By Lemma 3.3, $(G : C_G(A)) < 30$. Since A is self-centralising in H, and $H \subseteq N_G(A)$, we have $(N_G(A) : C_G(A))$ divisible by 4. Thus, $(G : C_G(A)) = 4n$ for some $n \leq 7$. To simplify notation, put $X = \mathrm{Core}_G(N_G(A))$ again. Then we have a natural injection $X/X \cap C_G(A) \to N_G(A)/C_G(A)$, and a subnormal sequence as in (15). First, suppose $(G : C_G(A)) \leq 12$. Then $(G : X) \leq 6$, and it is easy to see from (15) that it is impossible to avoid the contradiction that $G^{(3)} = H \subseteq C_G(A)$.

Next, suppose $(G : C_G(A)) = 16$. From (15), the only way to avoid the contradiction that $H \subseteq C_G(A)$ is to assume that $(G : N_G(A)) = 4$ and that $G/X \cong S_4$. One may check that $l(S_4) = l(\mathrm{id}, S_4) = \frac{5}{12}$. It is at this point that we use α-invariance of A. Since A is α-invariant, so is X, so we can apply Lemma 1.2 to obtain $l(G) \leq l(X)l(G/X) = \frac{5}{12}l(X)$. But $H \subseteq X$, so $l(X) \leq l(H) = \frac{5}{8}$. Therefore, $l(G) \leq \frac{5}{8} \times \frac{5}{12} < \frac{4}{15}$, a contradiction.

Next, suppose $(G : C_G(A)) = 4p$, where $p = 5$ or 7. In either case, either $C_G(A) \lhd G$ - which immediately gives the usual contradiction that $H \subseteq C_G(A)$ - or $(G : N_G(A)) = p$, $X/X \cap C_G(A)$ is abelian and G/X is isomorphic to a soluble subgroup of S_p, of order divisible by p. It is a fact (perhaps well-known) that this implies G/X is metabelian, so once again we have the contradiction, using (15), that $H \subseteq C_G(A)$. For the convenience of the reader, we give a proof of this 'fact' :

Fact 3.5. *Let G be a soluble subgroup of S_p, where p is a prime, of order divisible by p. Then G is a primitive permutation group. It is a semi-direct product of C_p by C_q for some integer q dividing $p - 1$. In particular, G is metabelian.*

PROOF OF FACT : The reader is referred to Chapter 10 of [8]. A priori G, as a subgroup of S_p, is either intransitive, imprimitive or primitive. In the first two cases, p cannot divide $\mid G \mid$. So G is primitive. By Theorem 10.5.21 of [8], G has a unique minimal normal, non-identity subgroup H of order p. By Theorem 10.3.5 of [8], H is its' own centraliser in S_p, hence G is an extension of H by a subgroup of $\mathrm{Aut}(H) \cong C_{p-1}$. Since $(p, p-1) = 1$, this extension is a semi-direct product, which completes the proof of our fact.

In order to eliminate *Case II*, it remains to deal with the possibility that $(G : C_G(A)) = 24$. If $N_G(A) \lhd G$, we immediately obtain, from (15), the contradiction that $H \subseteq C_G(A)$. So $(G : N_G(A)) = 3$ or 6.

First, suppose $(G : N_G(A)) = 6$. Since $X/X \cap C_G(A)$ is now abelian, G/X must be isomorphic to a soluble subgroup of S_6 of derived length at least 3. Let P be any such subgroup of S_6. If we can show that $l(P) < \frac{4}{15}/\frac{5}{8} = \frac{32}{75}$, then we can use the same arugment as above (when we had $(G : C_G(A)) = 16$) to obtain $l(G) < \frac{4}{15}$. We now do this.

The possibilities for P are discussed in the last paragraph of the Appendix. If P is intransitive then $P \cong S_4$ or $S_4 \times C_2$. But $l(S_4) = l(S_4 \times C_2) = \frac{5}{12} < \frac{32}{75}$, as desired.

If P is transitive, then P is isomorphic to a subgroup of either $(C_2 \times C_2 \times C_2) \rtimes S_3$ or $(S_3 \times S_3) \rtimes C_2$. The former group is isomorphic to $S_4 \times C_2$ (it has two subgroups isomorphic to S_4, one of which is the intersection with A_6, whereas the other contains odd permutations), and its' only non-metabelian subgroups are isomorphic to S_4 or $S_4 \times C_2$, which have already been dealt with above. Furthermore, one may check that every proper subgroup of $(S_3 \times S_3) \rtimes C_2$ is metabelian. It thus remains to show that if $P \cong (S_3 \times S_3) \rtimes C_2$, then $l(P) < \frac{32}{75}$.

P has a characteristic elementary abelian subgroup E of order 9 and $C_P(E) = E$. Now suppose there exists $\tau \in \mathrm{Aut}(P)$ such that $l(\tau, P) \geq \frac{32}{75}$. We apply an argument similar to that used in the proofs of Lemmas 2.2 and 3.2. Without going into too many details, let u be the number of cosets of E which consist entirely of elements of $I_P(\tau)$ and v the number of other cosets which intersect $I_P(\tau)$. Then, since $l(P/E) = l(D_4) = \frac{3}{4}$, we have the pair of inequalities

$$9u + 3v \geq (\frac{32}{75})72 \tag{19}$$

and

$$u + v \leq 6. \tag{20}$$

Combining these inequalities, we get $u \geq 3$. But, by (8), this contradicts the fact that E is self-centralising in P.

Finally, it remains to deal with the possibility that $(G : N_G(A)) = 3$. Since A has 3 maximal subgroups containing $Z(H)$, at least one of these is α-invariant. Pick one such and call it B. Then $C_G(B) \supset C_G(A)$ and $(G : C_G(B))$ divides 12. Let $Y = \mathrm{Core}_G(N_G(B))$. By considering the subnormal sequence

$$Y \cap C_G(B) \lhd Y \lhd G, \tag{21}$$

one quickly sees that the only way to avoid the contradiction that $H \subseteq C_G(B)$ is to have $(G : C_G(B)) = 12$ and $(G : N_G(B)) = 6$. Thus, as above, G/Y must be a soluble subgroup of S_6 of derived length at least 3, and satisfying $l(G/Y) > \frac{32}{75}$. As we have just shown, this is impossible.

Case III : This is very easy to deal with using the same types of arguments. By [4], $H/Z(H)$ is elementary abelian of order 2^6. H has many abelian subgroups of index 8, containing $Z(H)$. Let A be any one of these. By Lemma 3.3, $(G : C_G(A)) < 30$. But A is self-centralising in H and $H \subseteq N_G(A)$, so $(G : C_G(A)) \leq 24$ and is a multiple of 8. Now let B be any subgroup of A of index 4 containing $Z(H)$. Since $(H : C_H(B)) = 2$, it follows that $(C_G(B) : C_G(A))$ is divisible by 4. Therefore, $(G : C_G(B)) \leq 6$. Letting $X = \mathrm{Core}_G(N_G(B))$, and considering the subnormal sequence $X \cap C_G(B) \lhd X \lhd G$, we see that it is impossible to avoid the contradiction that $G^{(3)} = H \subseteq C_G(B)$.

This eliminates *Case III*, and completes the proof of Theorem 3.1.

Remark 3.6. Before finishing up, let's clarify what questions we have left open. Obviously, one would eventually like to obtain the precise value of l_n for all $n > 4$. This may be very difficult, so one may confine one's attention to the case $n = 5$, where I suspect it is not too hard to actually construct a group $G \in \Xi_5$ with $l(G)$ 'close' to $\frac{4}{15}$.

It is pretty clear that the arguments we have presented can be pushed to improve upon both Theorems 2.6 and 3.1, though it seems likely that some fresh ideas will be required to obtain significantly better results. For example, our whole strategy for obtaining a bound on $l(G)$, when $G \in \Xi_n$, was based on investigating the group $G^{(n-2)}$ - the other terms of the lower central series were completely ignored. In the cases we considered, this group always satisfied $l(G^{(n-2)}) > \frac{1}{2}$, so we could use the classification in [4] to obtain vital information about it. If we wish to utilise the solubility of G more comprehensively, it seems likely this will require a generalisation of the classification in [4]. Perhaps one can say something about groups $G \in \Xi_n$ satisfying $l(G) > l_{n+1}$?

Appendix

In this appendix, we will describe the sturcture of the group C of central automorphisms of H, when H is one of the groups $Q \circ Q$, $Q \circ D_4$ or $Q \circ Q \circ Z$, and Z is cyclic of order 4. In particular, we will show that in all cases, every soluble subgroup of C has derived length at most 4.

CASE I : $H \cong Q \circ Q$.

H is extraspecial so C is the full automorphism group of H. Its' structure is given by Winter's Theorem [9]. More explicitly, C is an extension of an elementary abelian normal subgroup E of order 16 by an orthogonal group O of order 72. The group E stabilises each of the six C_4 subgroups of H, whereas O is an extension of $S_3 \times S_3$ by C_2, where $S_3 \times S_3$ stabilises each quaternion group and permutes the three C_4 subgroups which it contains as in the group G_2, while C_2 interchanges the two quaternion groups. In particular, O is soluble of length 3, and hence C is soluble of length 4.

CASE II : $H \cong D_4 \circ Q$.

Once again, H is extraspecial, so we may refer to [9]. This time C is an extension of an elementary abelian subgeoup E of order 16 by an orthogonal group $O \cong S_5$. More explicitly, the group E stabilises each of the five Klein-4 subgroups of H, whereas the group O permutes them as S_5. Now every soluble subgroup of S_5 has derived length at most 3 - indeed, this is true of every soluble subgroup of S_6, as will be proven in CASE III below. Hence, every soluble subgroup of C has derived length at most 4, as required.

CASE III : $H \cong Q \circ Q \circ Z \cong D_4 \circ Q \circ Z$.

In this case, $C = C_{\mathrm{Aut}(H)}(Z)$. H is not extraspecial, so we shall obtain the structure of C from scratch. We claim that C is an extension of an elementary abelian group E of order 16 by S_6.

Note that C must contain each of the groups found in CASES I and II. We remark first that the group $Q \circ Z$ has one quaternion subgroup and 3 dihedral subgroups; also $Q \circ Q$ has 2 quaternion subgroups, each of which is the centraliser of the other; finally, $D_4 \circ Q$ has 5 elementary abelian subgroups of order 4, no two of which commute; hence $D_4 \circ Q$ has 10 dihedral subgroups, and the centraliser of each is a quaternion group.

Now the group H has 30 subgroups K of order 4 such that $K \cap Z = H'$, and 15 of them are cyclic while the other 15 are Klein-4 groups. Given such a subgroup K, of the 30 subgroups of order 4 whose intersection with Z is H', 14 commute with K (including K itself) and 16 do not. Moreover, of the latter, 8 are cyclic and 8 elementary abelian - this holds whether K itself is cyclic or elementary abelian. Now H has 140 subgroups L of order 8 such that $L \cap Z = H'$, and using the facts above we see that 20 of them are quaternion groups, 60 are dihedral and the other 60 are abelian. Also, the centraliser of each non-abelian L is isomorphic to $Q \circ Z$. Finally, there are 16 extrasepcial subgroups H_0 of order 32 such that $H = H_0 \circ Z$, and the above remarks about the subgroups L, together with the properties of $Q \circ Z$, $Q \circ Q$ and $D_4 \circ Q$, can be used to see that 10 of these subgroups H_0 are isomorphic to $Q \circ Q$, while 6 are isomorphic to $D_4 \circ Q$. The latter 6 subgroups are clearly permuted transitively by C. On the other hand, the stabiliser of each is isomorphic to $\mathrm{Aut}(D_4 \circ Q)$ hence, by CASE II above, an extension of an elementary abelian E of order 16 by S_5.

Thus C is an extension of E by S_6, as claimed.

It now remains to prove that every soluble subgroup of C has derived length at most 4, and for this it suffices to show that every soluble subgroup of S_6 has derived length at most 3. Let P be a soluble subgroup of S_6 which is not metabelian.

If P is intransitive then, using Theorems 10.1.8 and 10.5.21 of [8], we quickly deduce that $P \cong S_4$ or $S_4 \times C_2$ or a transitive subgroup of S_5. In the first two cases, it is immediate that $P^{(3)} = 1$. In the third case, since P is transitive in S_5, its' order must be divisible by 5. But then Fact 3.5 implies that P is metabelian, a contradiction.

If P is transitive, the P cannot be primitive, by Theorem 10.5.21 of [8]. Therefore, P is imprimitive. By Theorem 10.5.5 of [8], P must be isomorphic to a subgroup of either $(C_2 \times C_2 \times C_2) \rtimes S_3$ or $(S_3 \times S_3) \rtimes C_2$. But each of these groups is soluble of length 3, so we're done.

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