In class (Sats 1.1 and exercise 1.1) we proved the case $n = 2$ of the following well-known theorem

The AMGM inequality *Let* $a_1, a_2, ..., a_n$ *be positive real numbers. Then*

$$\frac{1}{n}(a_1 + \cdots + a_n) \geq \sqrt[n]{a_1 \cdots a_n}, \tag{1}$$

with equality if and only if $a_1 = a_2 = ... = a_n$.

There are many different proofs out there of this result and various generalisations of it, and every so often a new one appears in some popular mathematics journal. Here is one fairly short proof, taken from the book 'A course of pure mathematics', by G.H. Hardy.

PROOF : Denote the geometric mean (rhs of (1)) by G. Let a_r and a_s be the greatest and least of the a's respectively (if there are several of either, choose any one arbitrarily). If we replace a_r, a_s by

$$a_r' := G \qquad \text{and} \qquad a_s' := \frac{a_r a_s}{G},$$

then the rhs of (1) is unchanged, whereas since

$$(a_r' + a_s') - (a_r + a_s) = \frac{1}{G}(a_r - G)(a_s - G) \leq 0,$$

the lhs of (1) is not increased, and is in fact decreased unless $a_r = a_s = G$, which is the case if and only if all $a_i = G$.

Now repeat this procedure until each of $a_1, ..., a_n$ has been replaced by G - at most n steps are needed. At the end, the arithmetic mean (lhs of (1)) has become equal to G, so at the beginning it must have been bigger, unless all $a_i = G$ at the outest. This completes the proof !