In exercise 4.16, you proved by induction on n that

Theorem 1 *For each $n \geq 1$,*

$$\prod_{k=1}^{n} \frac{2k-1}{2k} \leq \frac{1}{\sqrt{3n+1}}. \tag{1}$$

It follows immediately that

$$\lim_{n\to\infty} \sqrt{n} \left(\prod_{k=1}^{n} \frac{2k-1}{2k} \right) \leq \frac{1}{\sqrt{3}}, \tag{2}$$

if the limit exists (by no means obvious). In fact, the limit does exist and we have the following result :

Theorem 2

$$\lim_{n\to\infty} \sqrt{n} \left(\prod_{k=1}^{n} \frac{2k-1}{2k} \right) = \frac{1}{\sqrt{\pi}}. \tag{3}$$

PROOF : Note that

$$\prod_{k=1}^{n} 2k = 2^n n!$$

Multiplying the product in (2) above and below by this number one sees that

$$\prod_{k=1}^{n} \frac{2k-1}{2k} = \frac{(2n)!}{2^{2n}(n!)^2}. \tag{4}$$

The proof is now completed by applying a famous formula due to Stirling. I know of no easy proof of this result, so I will just state it. In fact, all known proofs seem to use techniques from a branch of mathematics called *complex analysis.* which is the theory of differentiable functions of one complex variable. Maybe you'll take the course some time !

Stirling's formula

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\; n^n e^{-n}} = 1. \tag{5}$$

Now back to the proof of Theorem 2. From (4) and (5) we obtain

$$\text{LHS of ( )} = \lim_{n\to\infty} \sqrt{n} \cdot \frac{\sqrt{2\pi(2n)}\; (2n)^{2n} e^{-2n}}{2^{2n} \left(\sqrt{2\pi n}\; n^n e^{-n}\right)^2} = \frac{1}{\sqrt{\pi}},$$

as required.