Extra material : Day 10

The extra stuff today consists of a few remarks about counting trees.

DEFINITION : Let G_1, G_2 be simple graphs. An *isomorphism* from G_1 to G_2 is a bijection

$$\phi : V(G_1) \to V(G_2)$$

such that, for any two vertices v, w of G_1, $\{v, w\}$ is an edge in G_1 if and only if $\{\phi(v), \phi(w)\}$ is an edge in G_2.

If there exists such an isomorphism we say that G_1 is *isomorphic* to G_2. It's an easy exercise to check that 'being isomorphic to' is an equivalence relation on graphs, and hence we can talk about *isomorphism classes* of graphs. Informally, two graphs are isomorphic if they have the same number, say n, of vertices, and you can label the vertices of each graph from 1 to n in such a way that the two graphs then have the same edges.

REMARK 1 : As far as I know, no nice formula exists for the number of isomorphism classes of simple graphs on n vertices. In fact, even if we restrict our attention to trees, there is no such formula. This is maybe not surprising given the next remark -

REMARK 2 : The *graph isomorphism problem* is the algorithmic problem of deciding whether two arbitrarily given graphs are isomorphic. I believe the current status of this problem is that it is in NP (i.e.: no polynomial-time algorithm is known), but it is an open question as to whether it is NP-complete. As such, it has a similar status to the *integer factorisation problem.*

DEFINITION : Let $n > 0$. A *labelled graph* on n vertices is a pair (G, α), where G is a graph on n vertices and α is a bijection from $V(G)$ to $\{1, ..., n\}$. For obvious reasons, the map α is called the *labelling* of the graph G.

DEFINITION : Two labelled, simple graphs (G_1, α_1) and (G_2, α_2) are said to be *isomorphic* (as labelled graphs) if they have the same number, say n, of vertices, and the map

$$\alpha_2^{-1} \circ \alpha_1 : V(G_1) \to V(G_2)$$

is an isomorphism of (ordinary) graphs.

If we restrict our attention to trees, then this time there is the following surprisingly simple fact :

Theorem (Cayley) *The number of isomorphism classes of labelled trees on n vertices is n^{n-2}.*

I don't know of any 'really' simple proof of this theorem. The book 'A Course in Combinatorics' by J.H. van Lint and R.M. Wilson has three proofs !!

I also made a remark in class to the effect that the number of 'plane rooted trees' on n vertices is just the Catalan number C_{n-1}. Here we are distinguishing between any two trees which only differ by a symmetry of the plane. There is an explicit 1-1 correspondence between such trees on n vertices and Dyck paths of length $2n-2$, namely : take a walk around the tree, from 'left to right', starting and finishing at the root. Since there are $n-1$ edges and each edge is passed over twice, your walk consists of $2n-2$ steps. The corresponding Dyck path has an up-step (resp. down-step) for every step in your walk which takes you further away from (resp. nearer) the root. The path hits the x-axis every time you arrive back at the root during your walk.