Extra material : Day 5

The extra material in this lecture was about the so-called *Catalan numbers.*

DEFINITION : Let n be a non-negative integer. A *Dyck path of length* $2n$ is a path in the xy-plane from $(0,0)$ to $(2n,0)$ consisting of $2n$ steps, each of the form

$$(x,y) \mapsto (x+1, y \pm 1),$$

which in addition never goes below the x-axis.

DEFINITION : Let $n \geq 0$. The n^{th} *Catalan number*, denoted C_n, is defined to be the number of Dyck paths of length $2n$.

Theorem 1 *The Catalan numbers satisfy the following recurrence relation*

$$C_0 = 1, \tag{1}$$

$$C_n = \sum_{m=1}^{n} C_{m-1} C_{n-m}, \qquad \forall\, n \geq 1. \tag{2}$$

PROOF : (1) is obvious. For (2) we observe that $C_{m-1}C_{m-n}$ is the number of Dyck paths pf length $2n$ which first intersect the x-axis at $(2m,0)$.

Theorem 2

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

PROOF : We work with the generating function for the sequence (C_n), i.e.: the function

$$F(x) = \sum_{n=0}^{\infty} C_n x^n.$$

Using (1) and (2) we have that

$$x \cdot [F(x)]^2 = (xF(x)) \cdot F(x) = \left(\sum_{m=1}^{\infty} C_{m-1} x^m \right) \cdot \left(\sum_{t=0}^{\infty} C_t x^t \right)$$
$$= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} C_{m-1} C_{n-m} \right) x^n$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} C_n x^n \\
&= F(x) - C_0 \\
&= F(x) - 1,
\end{aligned}
$$

i.e.:

$$x[F(x)]^2 = F(x) - 1. \tag{3}$$

We may consider (3) as a quadratic equation for $F(x)$, and hence there are two possible solutions, namely

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Since $F(0) = C_0 = 1$, the correct solution must be to take the minus sign. We conclude that

$$F(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

To expand this in a power series, we use the binomial theorem for exponent 1/2 (which can be proven using complex analysis : note that one should not use the proof given in my solution to exercise C on homework 2, since that used the formula for C_n already, which means we'd be arguing in a circle !). So let's get to it !

$$
\begin{aligned}
F(x) &= \frac{1 - \sqrt{1-4x}}{2x} \\
&= \frac{1}{2x}\left[1 - (1-4x)^{1/2}\right] \\
&= -\frac{1}{2x}\sum_{n=1}^{\infty} \binom{1/2}{n} (-4x)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 4^{n+1}}{2} \binom{1/2}{n+1} x^n.
\end{aligned}
$$

So it remains to prove that, for every integer $n \geq 0$,

$$\frac{(-1)^n 4^{n+1}}{2} \binom{1/2}{n+1} = \frac{1}{n+1} \binom{2n}{n}.$$

To see how to prove this identity, follow the beginning of my solution to exercise C on homework 2.