Extra material : Day 7

Theorem (Dirac) *Let G be a simple graph on n vertices in which every vertex has degree at least $n/2$. Then G has a Hamilton cycle.*

PROOF : Let G be a counterexample with the largest possible number of edges - we shall derive the contradiction that G contains a Hamilton cycle after all. By maximality of $|E(G)|$, if x and y are any two non-adjacent vertices in G, then addition of the edge $\{x, y\}$ would create a Hamilton cycle in G. In other words, there must exist a path from x to y in G passing through all the vertices. Denote the path as

$$y = x_1 \to x_2 \to \cdots \to x_{n-1} \to x_n = z.$$

We consider two sets S and T, defined as follows :

$$S = \{i : 1 \leq i \leq n-1 \text{ and } \{y, x_{i+1}\} \in E(G)\},$$
$$T = \{i : 1 \leq i \leq n-1 \text{ and } \{x_i, z\} \in E(G)\}.$$

On the one hand, both S and T are subsets of $\{1, ..., n-1\}$, so

$$|S \cup T| \leq n - 1. \tag{1}$$

On the other hand,

$$|S| = \deg(y) \geq n/2,$$
$$|T| = \deg(z) \geq n/2,$$

which implies that

$$|S| + |T| \geq n. \tag{2}$$

From (1) and (2), we can deduce that $S \cap T$ is non-empty. Let $i \in S \cap T$. Then the following is a Hamilton cycle in G :

$$y \to x_2 \to \cdots x_i \to z \to x_{n-1} \to \cdots \to x_{i+1} \to y.$$

With that, the proof of the theorem is complete.