Extra material : Day 8

The extra stuff today consists of some brief remarks about planar graphs.

DEFINITION : A graph is said to be *planar* if it can be drawn in the plane without any two edges crossing. If the graph is actually drawn in such a way then it is called *plane*.

There is a nice simple characterisation of planarity due to Kuratowski. First, we need a definition :

DEFINITION : Let G, H be graphs. G is said to contain H as a *minor* if H can be obtained from G by a number of edge-contractions and deletions.

Theorem (Kuratowski) *A graph is planar if an only if it does not contain either K_5 or $K_{3,3}$ as a minor.*

Theorem (4-colour theorem) *If G is a planar graph, then $\chi(G) \leq 4$.*

This famous theorem was first proven in the mid '70s. The main reason it is so famous is that it was the first long-standing open problem which was solved with substantial help from computers. The idea of the proof was to exhibit a finite collection of 'minimal possible counterexamples', that is, a finite collection of planar graphs such that, if each of these were 4-colourable, then it would follow that all planar graphs were. Such a list, consisting of about 10,000 graphs, was produced. And then the computers were called in to check that each of these graphs was indeed 4-colourable. In the '90s a simplified version of the proof was obtained by Seymour, Robertson, ? and ??, which reduced the number of 'test graphs' to about 600.

Theorem (Euler) *Let G be a plane graph. Let V denote the number of vertices in G, E the number of edges and R the number of regions into which G divides the plane, where we include the unbounded region outside G as one such. Then*

$$V - E + R = 2.$$

PROOF : I didn't go through it, but the standard method is to use induction on the number of edges in G.

REMARK : Let X be any compact, smooth surface in $\mathbf{R}^3$. Let G be any graph drawn on this surface in such a way that no two edges cross and such that G divides the surface into simply-connected regions. If we compute $V - E + R$, then the result depends only on X, but not on G. This number is called the *Euler characteristic* of the surface X and denoted $\chi(X)$. See a textbook on Differential Geometry for more.