## Extra material : Day 9

**Theorem (Mantel 1907)** *Let $G$ be a simple graph on $n$ vertices. If $G$ contains more than $\lfloor \frac{n^2}{4} \rfloor$ edges, then $G$ contains a triangle.*

Remark : If $n = 2k$ is even then the complete bipartite graph $K_{k,k}$ has $n$ vertices, $n^2/4$ edges and no triangles. If $n = 2k+1$ is odd then the complete bipartite graph $K_{k,k+1}$ has $n$ vertices, $\lfloor n^2/4 \rfloor$ edges and no triangles. So Mantel's Theorem is the best-possible result.

Proof of Theorem : We assign to each vertex $v$ of $G$ a non-negative real number $q_v$ such that

$$\sum_{v \in V} q_v = 1.$$

We refer to $q_v$ as the 'charge' on the vertex $v$. We'll be interested in how to distribute the charge so as to maximise the quantity

$$S := \sum_{\{v,w\} \in E(G)} q_v q_w.$$

Let $x, y$ be any two non-adjacent vertices in $G$. Let $X$ (resp. $Y$) denote the total charge on the vertices adjacent to $x$ (resp. $y$). WLOG, suppose $X \geq Y$. Suppose we move all the charge on $y$ to $x$. Then the change in $S$ is given by

$$\Delta S = (X - Y) q_y \geq 0.$$

It follows that $S$ attains a maximum when all the charge is concentrated on a complete subgraph of $G$. Now if $G$ contains no triangles, then the only complete subgraphs in $G$ consist of a single edge. So $S$ is maximised when we have charges $q$ and $1 - q$ on two adjacent vertices, for some $q \in [0, 1]$. Then

$$S = S(q) = q(1 - q).$$

This one-variable function attains its' maximum at $q = 1/2$ and so the maximum of $S$ is $1/4$.

Suppose we were instead to distribute the charge uniformly over all $n$ vertices. Then each vertex gets charge $1/n$ and so in this case

$$S = \frac{1}{n^2}|E(G)|.$$

Hence, if $G$ contains no triangles, it follows that

$$\begin{aligned} \frac{1}{n^2}|E(G)| &\leq 1/4 \\ \Rightarrow |E(G)| &\leq \frac{n^2}{4}, \end{aligned}$$

as required.