MAN 240 : Diskret matematik

Tentamen 040603

Lösningar

F.1 (i) First choose the three positions for the zeroes $\Rightarrow \binom{16}{3}$ choices.

Next choose the four positions for the ones $\Rightarrow \binom{13}{4}$ choices.

Now you have 9 positions left, and each can be filled with any one of the 8 digits from $2-9$. Thus, by the multiplication principle, there are 8^9 choices for these remaining positions.

A final application of the multiplication principle then implies that the total number of possible credit card numbers is

$$\binom{16}{3} \cdot \binom{13}{4} \cdot 8^9.$$

(ii) Subtract from $n!$ the number of permutations which move at most two numbers. For $i = 0, 1, 2$ let A_i denote the number of permutations which move exactly i numbers. Then

$A_0 = 1$ (the identity permutation)
$A_1 = 0$ (can't move just one number)
$A_2 = \binom{n}{2}$ (choose a pair of nos. to interchange).

Hence, the answer is $n! - 1 - \binom{n}{2}$.

F.2 (i) The graph has Hamilton cycles, for example

$$A \to B \to E \to H \to K \to I \to J \to G \to F \to C \to D \to A.$$

(ii) $\chi(G^*) \geq 3$ since G^* contains many triangles. On the other hand, if we apply the greedy algorithm to the nodes ordered alphabetically, then we get a 3-coloring, namely (the colors are $1, 2, 3$)

A	1	G	1
B	2	H	2
C	3	I	3
D	2	J	2
E	1	K	1
F	2		

Hence $\chi(G^*) = 3$.

(iii) Use BFS, starting, say, from the vertex A, to build up the following sequence of edges in a MST :

$$\{A, D\}, \{A, B\}, \{B, E\}, \{E, C\}, \{C, F\},$$
$$\{C, G\}, \{G, J\}, \{J, I\}, \{I, K\}, \{I, H\} \text{ (or } \{J, H\}).$$

The total weight of this tree is $1 + 2 + 1 + 5 + 1 + 2 + 1 + 1 + 2 + 7 = 23$.

(iv) Apply Dijkstra's algorithm to build up the following tree

Step	Choice of edge	Labelling
1	AD	$D := 1$
2	AB	$B := 2$
3	BE	$E := 3$
4	AC/DC	$C := 8$
5	CF	$F := 9$
6	DG/CG	$G := 10$
7	EH	$H := 11$
8	GJ	$J := 11$
9	JI/EI	$I := 12$
10	IK	$K := 14$

Hence the shortest path from A to K has length 14. Depending on the choices you made in Steps 4,6,8, there are four possibilities for the shortest path, namely

$$A \to B \to E \to I \to K,$$
$$A \to D \to G \to J \to I \to K,$$
$$A \to D \to C \to G \to J \to I \to K,$$
$$A \to C \to G \to J \to I \to K.$$

(v) Starting with the null flow $f \equiv 0$, one can find the following sequence of f-augmenting paths from A to K (these are not the only choices, of course) :

$$\begin{aligned} A-C-E-H-K, &\quad \epsilon = 5, \\ A-B-E-H-K, &\quad \epsilon = 1, \\ A-C-G-I-K, &\quad \epsilon = 2, \\ A-C-F-E-H-K, &\quad \epsilon = 1, \\ A-D-G-J-K, &\quad \epsilon = 1. \end{aligned}$$

This yields the following maximal flow with $|f| = 10$

Edge	Flow	Edge	Flow	Edge	Flow
(A,B)	1	(C,G)	2	(G,I)	2
(A,C)	8	(D,G)	1	(G,J)	1
(A,D)	1	(F,E)	0	(I,H)	1
(C,B)	0	(G,F)	0	(J,I)	0
(D,C)	0	(E,H)	6	(H,K)	7
(B,E)	1	(E,I)	0	(I,K)	2
(C,E)	5	(F,I)	1	(J,K)	1
(C,F)	1				

The corresponding minimal cut is $S = \{A,B\}$, $T =$ rest of them. Its' capacity is given by

$$c(S,T) = c(B,E) + c(A,C) + c(A,D) = 1 + 8 + 1 = 10, \quad \text{v.s.v..}$$

F.3 This is Mantel's Theorem. See my extra lecture notes for Day 9.

F.4 To simplify notation, if v is a vertex of the graph G, let $G \backslash v$ denote the graph obtained by deleting v and all edges through it.

Now since G is connected it has a spanning tree T. Being a tree, T has at least 2 leaves (see exercise 8.5.2 in Biggs). Let v be a leaf of T. Then $T \backslash v$ is still connected and is a spanning tree for $G \backslash v$, which implies that $G \backslash v$ is also connected.

F.5 Everything is contained in my extra lecture notes for Day 5.

F.6 Let

$$F(x) = \sum_{n=0}^{\infty} u_n x^n$$

denote the generating function of the sequence (u_n). Let's rock !

$$\begin{aligned}
(1-4x-5x^2)F(x) &= (u_0 + u_1 x) - 4(u_0 x) + \sum_{n=2}^{\infty}(u_n - 4u_{n-1} - 5u_{n-2})x^n \\
&= (4-x) - 4(4x) + \sum_{n=2}^{\infty} 3^n x^n \\
&= 4 - 17x + \frac{9x^2}{1-3x} \\
&= \frac{60x^2 - 29x + 4}{1-3x}.
\end{aligned}$$

Since

$$1 - 4x - 5x^2 = (1+x)(1-5x),$$

we conclude that

$$F(x) = \frac{60x^2 - 29x + 4}{(1+x)(1-5x)(1-3x)}.$$

We seek a partial fraction decomposition

$$\frac{60x^2 - 29x + 4}{(1+x)(1-5x)(1-3x)} = \frac{A}{1+x} + \frac{B}{1-5x} + \frac{C}{1-3x}. \tag{1}$$

Clearing denominators, we have

$$60x^2 - 29x + 4 = A(1-5x)(1-3x) + B(1+x)(1-3x) + C(1+x)(1-5x).$$

Gathering coefficients, we get the following system of linear equations to solve

$$\begin{pmatrix} 15 & -3 & -5 \\ -8 & -2 & -4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 60 \\ -29 \\ 4 \end{pmatrix}.$$

After the usual Gauβ elimination and back substitution (I omit the details), we get the solution

$$A = \frac{31}{8}, \quad B = \frac{5}{4}, \quad C = -\frac{9}{8}.$$

Substituting into (1) and using the relation

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n,$$

we conclude that

$$F(x) = \frac{31}{8}\sum_{n=0}^{\infty}(-1)^n x^n + \frac{5}{4}\sum_{n=0}^{\infty} 5^n x^n - \frac{9}{8}\sum_{n=0}^{\infty} 3^n x^n.$$

After a little tidying up, it follows that

$$u_n = \frac{1}{8}\left(31 \cdot (-1)^n + 2 \cdot 5^{n+1} - 3^{n+2}\right).$$