**MAN 240 : Diskret matematik**

**Tentamen 230803**

**Lösningar**

**F.1** **(i)** Fix $k \geq 3$. We first compute the number of 7-digit decimals containing exactly $k$ 9:s. If the first digit is not a 9, then the number of possibilities is

$$\binom{6}{k} \cdot 8 \cdot 9^{6-k},$$

since there are $\binom{6}{k}$ possibilities for the positions of the 9:s, 8 options for the first digit and 9 for each of the remaining $6-k$ digits.

Similarly, if the first digit is a 9, then the number of possibilities is

$$\binom{6}{k-1} \cdot 9^{7-k}.$$

Summing over $k$, we thus find that the total number of 7-digit decimals containing at least three 9:s is

$$\sum_{k=3}^{7} \binom{6}{k} \cdot 8 \cdot 9^{6-k} + \binom{6}{k-1} \cdot 9^{7-k}.$$

This works out as 241065, though you weren't expected to compute this number.

NOTE : An alternative approach is to compute the number of 7-digit decimals containing at most two 9:s and subtract this from $9 \cdot 10^6$.

**(ii)** The answer is $8!/3!$, which is most easily explained as follows : there are $8!$ possible orderings of the 8 countries. Among all these orderings, each of the $3!$ 'suborderings' of Denmark, Italy and Liechtenstein appears equally often.

**F.2** **(i)** The graph has Hamilton cycles, for example

$$s \to a \to d \to h \to t \to i \to e \to b \to f \to g \to c \to s.$$

**(ii)** $\chi(G^*) \geq 3$ since $G^*$ contains a triangle (exactly one triangle in fact, namely $\Delta hit$). On the other hand, if we apply the greedy algorithm to the nodes ordered as $s, a, b, c, d, e, f, g, h, i, t$ then we get a 3-coloring, namely (the colors are $1, 2, 3$)

| | | | |
|---|---|---|---|
| s | 1 | f | 2 |
| a | 2 | g | 1 |
| b | 1 | h | 2 |
| c | 2 | i | 1 |
| d | 1 | t | 3 |
| e | 2 | | |

Hence $\chi(G^*) = 3$.

**(iii)** Use BFS, starting, say, from the vertex $s$, to build up the following sequence of edges in a MST :

$$\{s, c\}, \{c, g\}, \{f, g\}, \{f, i\}, \{i, h\},$$
$$\{d, h\}, \{b, f\}, \{a, b\}, \{i, t\}, \{b, e\}.$$

The total weight of this tree is $13 + 4 + 8 + 5 + 3 + 5 + 6 + 4 + 7 + 8 = 63$. Note that it is the unique spanning tree of this weight.

**(iv)** Apply Dijkstra's algorithm to build up the following tree

| Step | Choice of edge | Labelling |
|---|---|---|
| 1 | $sc$ | $c := 13$ |
| 2 | $sa$ | $a := 14$ |
| 3 | $cg$ | $g := 17$ |
| 4 | $ab$ | $b := 18$ |
| 5 | $ad$ | $d := 22$ |
| 6 | $bf$ | $f := 24$ |
| 7 | $be$ | $e := 26$ |
| 8 | $dh$ | $h := 27$ |
| 9 | $gt$ | $t := 29$ |

Hence the shortest path from $s$ to $t$ is $s \to c \to g \to t$ and has length 29. Note that this is the only path from $s$ to $t$ of length 29.

**(v)** Starting with the null flow $f \equiv 0$, one can find the following sequence of $f$-augmenting paths from $s$ to $t$ (these are not the only choices, of course) :

$$\begin{aligned} s-a-d-e-i-t, &\quad \epsilon=7, \\ s-c-b-f-g-t, &\quad \epsilon=6, \\ s-c-g-t, &\quad \epsilon=4, \\ s-c-b-e-i-h-t, &\quad \epsilon=3, \\ s-a-b-e-d-h-t, &\quad \epsilon=4, \\ s-a-d-h-t, &\quad \epsilon=1. \end{aligned}$$

This yields the following maximal flow with $|f| = 25$

| Edge | Flow | Edge | Flow | Edge | Flow |
|---|---|---|---|---|---|
| $(s,a)$ | 12 | $(b,f)$ | 6 | $(f,g)$ | 6 |
| $(s,c)$ | 13 | $(c,g)$ | 4 | $(i,h)$ | 3 |
| $(a,b)$ | 4 | $(d,e)$ | 3 | $(h,t)$ | 8 |
| $(c,b)$ | 9 | $(d,h)$ | 5 | $(i,t)$ | 7 |
| $(a,d)$ | 8 | $(e,i)$ | 10 | $(g,t)$ | 10 |
| $(b,e)$ | 7 | $(f,i)$ | 0 | | |

The corresponding minimal cut is $S = \{s,a\}$, $T =$ rest of them. Its' capacity is given by

$$c(S,T) = c(a,d) + c(a,b) + c(s,c) = 8 + 4 + 13 = 25, \quad \text{v.s.v..}$$

**F.3** Theorem 10.4 (resp. 17.4) in old (resp. new) Biggs.

**F.4** The optimal bound is $\chi(G) \leq 3$. If $G$ is a triangle, then the bound $\chi(G) = 3$ is attained. On the other hand, $\chi$ cannot be greater than 3. For, since $G$ is connected, it has a spanning tree $T$. Being a tree, we have $\chi(T) = 2$. Also, $T$ has $n$ vertices and $n-1$ edges, so $G$ consists of $T$ plus a single edge. We can now 3-color $G$ as follows : first, 2-color $T$. Let $\{x,y\}$ be the extra edge in $G$. If vertices $x$ and $y$ get different colors in the 2-coloring of $T$, then this is also a 2-coloring of $G$. Otherwise, change the color of $y$ say to the 3rd color available, and thus obtain a 3-coloring of $G$.

**F.5** Theorem 18.3 (resp. 25.3) in old (resp. new) Biggs.

**F.6** Let

$$F(x) = \sum_{n=0}^{\infty} u_n x^n$$

denote the generating function of the sequence $(u_n)$. Let's rock !

$$\begin{aligned}(1-3x-4x^2)F(x) &= (u_0+u_1x) - 3(u_0x) + \sum_{n=2}^{\infty}(u_n - 3u_{n-1} - 4u_{n-2})x^n \\ &= (2+3x) - 3(2x) + \sum_{n=2}^{\infty} 2^n x^n \\ &= 2 - 3x + \frac{4x^2}{1-2x} \\ &= \frac{10x^2-7x+2}{1-2x}.\end{aligned}$$

Since

$$1-3x-4x^2 = (1+x)(1-4x),$$

we conclude that

$$F(x) = \frac{10x^2-7x+2}{(1+x)(1-4x)(1-2x)}.$$

We seek a partial fraction decomposition

$$\frac{10x^2-7x+2}{(1+x)(1-4x)(1-2x)} = \frac{A}{1+x} + \frac{B}{1-4x} + \frac{C}{1-2x}. \tag{1}$$

Clearing denominators, we have

$$10x^2 - 7x + 2 = A(1-4x)(1-2x) + B(1+x)(1-2x) + C(1+x)(1-4x).$$

Gathering coefficients, we get the following system of linear equations to solve

$$\begin{pmatrix} 8 & -2 & -4 \\ 6 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 10 \\ -7 \\ 2 \end{pmatrix}.$$

After the usual Gau$\beta$ elimination and back substitution (I omit the details), we get the solution

$$A = \frac{19}{15}, \quad B = \frac{7}{5}, \quad C = -\frac{2}{3}.$$

Substituting into (1) and using the relation

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n,$$

we conclude that

$$F(x) = \frac{19}{15} \sum_{n=0}^{\infty} (-1)^n x^n + \frac{7}{5} \sum_{n=0}^{\infty} 4^n x^n - \frac{2}{3} \sum_{n=0}^{\infty} 2^n x^n.$$

It follows that

$$u_n = \frac{19}{15} \cdot (-1)^n + \frac{7}{5} \cdot 4^n - \frac{2}{3} \cdot 2^n.$$