MAN 240 (2004) : Inlämningsuppgift 1

(att lämnas in onsdag den 21 april)

1 (2.6.12 in old Biggs) A *lattice point* in 3-dimensional space is a point all of whose coordinates are integers. Show that if we are given nine lattice points then there is at least one pair for which the midpoint of the line segment joining them is also a lattice point.

(Hint : Pigeonhole principle).

2 (3.1.4 in old Biggs) Show that in any set of 10 people there are either four mutual friends or three mutual strangers.

3. The surface of a football consists of a network of 12 pentagons and 20 hexagons. At each 'node' two hexagons and a pentagon meet. Use this information to compute the number of nodes.

(Note : A solution which involves getting hold of a football and counting the nodes and/or looking up the answer on the internet is not valid !! The point is to get the answer by 'combinatorial reasoning').

4. How many different 'words' can you make from the letters of the word 'inlämningsuppgift' ?

(OBS! Only count those words in which you use all 17 letters).

5. How many poker hands give

(i) nothing
(ii) a pair
(iii) two pairs
(iv) three of a kind
(v) a straight
(vi) a flush
(vii) a full house
(viii) four of a kind
(ix) a straight flush
(x) a royal flush ?

6. You are faced with a class of 20 screaming children, so to shut them up you give them some candy. Say you have 70 pieces of candy and you want to make sure each child gets at least 2 pieces (since you're a sadist, you let them fight over the rest !!). How many ways can the candy be divided up ?

7. Simplify as much as possible the expression

$$\sum_{k=0}^{n} k \cdot \binom{n}{k}.$$

(N.B.: For full points, you will avoid all use of the formula for binomial coefficients).

8. Prove the following identities 'combinatorially' (i.e.: by showing that both sides count the same thing in two different ways and thereby avoiding use of the formula for binomial coefficients) :

$$\binom{n}{k} \cdot \binom{k}{i} = \binom{n}{i} \cdot \binom{n-i}{k-i},$$

$$n \cdot \binom{2n}{n} = (n+1) \cdot \binom{2n}{n+1}.$$

(In the first identity, $n \geq k \geq i$ are non-negative integers).

9. Let X be a finite set. Prove 'combinatorially' that the number of subsets of X consisting of an odd number of elements equals the number of subsets consisting of an even number of elements.

(What I have in mind here by a 'combinatorial' proof is that you describe an explicit 1-1 correspondence between the odd and the even subsets).

10. Let n be a positive integer. As I remarked in class, it is not clear whether knowledge of both n and $\phi(n)$ always leads to a quick procedure for factorising n (though this is generally believed to be the case). Show, however, that a simple such procedure exists when n is known to be the product of two distinct primes p and q.

11. Find a formula (in terms of binomail coefficients) for the number of integer solutions to the equation

$$a + b + c + d = 27,$$

where $a \leq 7$, $b \leq 10$, $c \leq 15$ and $d \leq 6$.

12. Let $k \geq 1$ be an integer. A set A of positive integers is said to be *k-sum-free* if there are no solutions in A to the equation

$$kx = y + z.$$

(i) Let $k = 1$. Here the term *sum-free* is used.

(a) For each integer $n > 0$ find, with proof, the largest possible size of a sum-free subset of $[1, n]$.

(b) Give, for each n, two examples of sum-free subsets of $[1, n]$ of maximal size.

***(ii)** Let $k = 3$. Same exercise as part **(i)(a)**.
****(iii)** Let $k = 4$. Same exercise as part **(i)(a)**.
*****(iv)** Let $k = 2$. A set is said to *avoid arithmetic progressions* if it contains no solutions to the equation

$$2x = y + z$$

other than solutions where $x = y = z$. Now solve the same problem as in part **(i)(a)**.

Further exercises from last year (not part of homework !!)

13. Without using induction on n, find and prove a formula for the sum of the first n odd positive integers.

14. A collection $\mathcal{C}$ of subsets of $\{1, ..., n\}$ is said to be intersecting if

$$A \cap B \neq \phi$$

for each pair of subsets from $\mathcal{C}$. Find (with proof) the maximum possible number of subsets in such a collection (as a function of n).

15. Let $k \leq l \leq n$ be positive integers. Let A, B be subsets of $\{1, ..., n\}$ of size k, l respectively. A is said to *cover* B if $A \subseteq B$.

Find (with proof) the smallest possible size of a collection of 2-element subsets of $\{1, ..., n\}$ which covers all 3-element subsets of $\{1, ..., n\}$.

16. Consider the map $\pi : \mathbf{N} \to \mathbf{N}$ defined inductively as follows :

(i) $\pi(1) = 1, \ \pi(2) = 2$,

(ii) for each $k \geq 3$, $\pi(k)$ is the smallest number which does not appear among $\pi(1), ..., \pi(k-1)$ such that, for no $1 \leq i \leq k/2$ is it the case that

$$\pi(k) - \pi(k-i) = \pi(k-i) - \pi(k-2i).$$

(Note : For obvious reasons, the map π is said to 'avoid arithmetic progressions'. The first few terms in the sequence $(\pi(n))$ are 1,2,4,3,5,6,8,7,10,9,13,...).

***(i)** Show that π is a bijection from $\mathbf{N}$ to $\mathbf{N}$.

(By definition, π is injective. The hard part is to show it is surjective).

***(ii)** Show that, in fact, for all positive integers n,

$$\frac{1}{4} \leq \frac{\pi(n)}{n} \leq \frac{3}{2}.$$

*****(iii)** Prove or disprove the conjecture that

$$\lim_{n \to \infty} \frac{\pi(n)}{n}$$

exists and equals 1.

* denotes an exercise which I consider somewhat more difficult than the others.

** denotes an exercise which I would consider a lot more difficult than the others.

*** denotes an exercise which, if you solve it, you will get VG on the course without having to do any more work. In fact you'll get a Ph.D. and maybe even become famous !!