MAN 240 (2004) : Inlämningsuppgift 2

1 (12.2.5(ii) in old Biggs) Prove that, for all $n \geq 0$,

$$f_n f_{n+2} = f_{n+1}^2 + (-1)^n,$$

where f_n is the n:th Fibonacci number.

2. Let $(u_n)_{n=0}^{\infty}$ be a sequence of numbers. The *exponential generating function* of the sequence is the power series

$$E(x) = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!}.$$

(i) Suppose the sequence (u_n) satisfies the recurrence relation

$$au_{n+2} + bu_{n+1} + cu_n = f_n,$$

where (f_n) is some 'known' sequence. Show that in that case the e.g.f. satisfies the differential equation

$$aE''(x) + bE'(x) + cE(x) = f(x),$$

where

$$f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}.$$

(Note : The point of this exercise is to show those of you familiar with differential equations the explicit connection between linear recurrence relations with constant coefficients and differential equations of the same type).

(ii) Let d_n denote the number of derangements of $\{1, ..., n\}$ as usual. In class we proved that

$$d_n = nd_{n-1} + (-1)^n, \qquad n \geq 2. \tag{1}$$

Now use (1) to prove that the exponential generating function of the sequence d_n (we define $d_0 = 1$) is

$$E(x) = \frac{e^{-x}}{1-x}.$$

From this, recover the usual explicit formula for d_n (which we also derived in class).

3 (see 12.7.4 in old Biggs) A triple (a, b, c) of integers with $a < b < c$ and $b - a = c - b$ is called an *arithmetic progression (AP)*. Let L_n denote the number of AP:s in $\{1, ..., n\}$. Prove that

$$L_{2k+1} = L_{2k} + k,$$

and prove a similar formula for L_{2k}. Hence deduce an explicit formula for L_n.

Now reprove this formula DIRECTLY, i.e.: without using the recurrence relation.

4 (see 12.7.11 and 18.7.17 in old Biggs) Let $\lambda(n, k)$ denote the number of k-element subsets of $\{1, ..., n\}$ containing no two consecutive integers.

(i) Show that

$$\lambda(n, k) = \lambda(n-2, k-1) + \lambda(n-1, k). \tag{2}$$

(ii) Hence, by considering the generating functions

$$F_k(x) = \sum_{n=0}^{\infty} \lambda(n, k) x^n,$$

prove that

$$\lambda(n, k) = \binom{n-k+1}{k}. \tag{3}$$

(iii) Now reprove the formula (3) DIRECTLY, i.e.: without using the recurrence relation (2).

5 (18.3.3 in old Biggs) Let a_n be the coefficient of x^n in the power series $(1 - x - x^2)^{-1}$. Prove that

$$a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}.$$

6 (18.6.2 in old Biggs) Let q_n be the number of words of length n in the alphabet $\{a, b, c, d\}$ which contain an odd number of b:s. Show that, for all $n \geq 0$,

$$q_{n+1} = 4^n + 2q_n.$$

Hence show that

$$q_n = \frac{1}{2}\left(4^n - 2^n\right).$$

7. Let $p(n,k)$ denote the number of partitions of the positive integer n into exactly k parts. Explain why

$$p(n,k) = p(n-1,k-1) + p(n-k,k).$$

8. Prove that, for all $n \geq 0$,

$$\frac{-\binom{1/2}{n+1}(-4)^{n+1}}{2} = \frac{1}{n+1}\binom{2n}{n}.$$

(Hint : Use the formula for generalised binomial coefficients).

9. Let n be a positive integer. A permutation $\pi : \{1,...,n\} \to \{1,...,n\}$ is said to be 132-*avoiding* if there does not exist any triple (i,j,k) such that

$$i < j < k \leq n,$$
$$\pi(i) < \pi(j) > \pi(k) > \pi(i).$$

Let A_n denote the number of 132-avoiding permutations of $\{1,...,n\}$. Show that $A_n = C_n$.

(Hint : Show that A_n satisfies the same recurrence relation as C_n. An alternative solution would be very interesting !)

10. Let B_n denote the number of sequences $a_1a_2\cdots a_n$ of n positive integers which are non-decreasing, i.e.:

$$a_i \leq a_{i+1}, \quad i = 1,...,n-1,$$

and such that

$$a_i \leq i, \quad i = 1,...,n.$$

For example, $B_3 = 5$ since there are the followinf five allowed sequences of length three :

111 112 113 122 123

Prove that $B_n = C_n$ for all n.

(Hint : Either prove the recurrence relation or describe an explicit 1-1 correspondence between such sequences and Dyck paths).

11. Let $n \geq 1$ and let D_n denote the number of n-tuples $(x_1, ..., x_n)$ of integers which satisfy

$$0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq n,$$

and

$$n+1 \text{ divides } \sum_{i=1}^{n} x_i.$$

Show that $D_n = C_n$ for all $n \geq 1$.

(Hint : Prove directly that $D_n = \frac{1}{n+1}\left(\begin{array}{c} 2n \\ n \end{array}\right)$. An alternative approach would be very interesting !)

12 (5.1.3 in old Biggs). Prove that, for any $n, k \geq 0$,

$$S(n,k) = \sum_{r=0}^{n-1} \left(\begin{array}{c} n-1 \\ r \end{array}\right) S(r, k-1).$$