MAN 240 (2004) : Inlämningsuppgift 3

In the following, to translate from old Biggs to new Biggs, replace chapters 8,9,10,11,12 by chapters 15,16,17,18 and 19. The term *graph* refers to simple graphs unless otherwise stated.

1. Get your hands on an up-to-date map of Europe. To have something to compare with, see for example
http://www.wunderground.com/global/Region/EU/Temperature.html

Make a graph G whose nodes are the countries of Europe, with an edge between every pair of countries which share a land border. Do not include countries of the former Soviet Union.

(a) Order the countries alphabetically according to their Swedish names, and then color G using the greedy algorithm. How many colors are used ?
(b) Do the same in English. How many colors used this time ?
(c) What is $\chi(G)$? Give at least two different reasons.

(Note : Don't forget to include teeny weeny countries like Lichtenstein, Andorra, San Marino, Vatican City and Monaco !!).

2. Exercises 8.8.5 and 8.8.6.

3 (a) How many pairwise non-isomorphic trees on 7 vertices are there ? Draw them all.
(b) According to Cayley's theorem, there are 16 labelled trees on 4 vertices. Draw them all !

4. Let G be any graph. Prove that either G or its' complement must be connected.

5. Let G be a directed graph without a directed cycle. Prove that G is a network, i.e.: has a source and a sink.

6. Let X be any compact 2-D surface in $\mathbf{R}^3$. Explain why $\chi(X) = 2 - 2g$, where g is the number of 'holes' in X.

7. Let P denote the Petersen graph.

(a) Give an explicit isomorphism between the usual pentagonal representation of P and the hexagonal one in exercise 8.8.3.
(b) Indicate cycles of lengths 5,6,8 and 9 in P.
(c) Show that it is not possible to edge-color P with 3 colors.
(d) Solve exercise 10.7.1.
(e) Hence, or otherwise, deduce that P has no Hamilton cycle.

8. Refer to the network in Fig. 12.3.

(a) Take away all the arrows and find a minimal weight spanning tree in the resulting undirected graph.
(b) Find a shortest path from s to t.
(c) Find a maximal flow from s to t.

9 (a) Give an example to show that the claim of exercise 8.8.22 is false.
(b) On the other hand show that, for any graph G on n vertices,

$$\chi(G) + \chi(\overline{G}) \leq n + 1.$$

10. Investigate which n-tuples $(d_1, ..., d_n)$ of positive integers can be the degrees of the vertices in a simple (resp. multi-) graph on n vertices. In this connection, solve exercise 8.8.17.

11. The following is a famous theorem :

Turan's theorem : Let n, p be positive intgers. Let t, r be such that

$$n = t(p-1) + r \quad \text{and} \quad 1 \leq r \leq p-1.$$

Put

$$M(n,p) := \frac{p-2}{2(p-1)}n^2 - \frac{r(p-1-r)}{2(p-1)}.$$

Then any simple graph on n vertices with more than $M(n,p)$ edges contains a copy of K_p.

(a) Give an example of a graph with n vertices and $M(n,p)$ edges which contains no K_p.

(b) Show that the argument we used to prove Mantel's theorem (the special case $p = 3$) gives, instead of exactly $M(n, p)$, the slightly weaker upper bound

$$|E(G)| \leq \frac{p-2}{2(p-1)} n^2.$$

***(c)** If you're feeling lucky, prove the theorem !!

12. Given 12 coins, one of which is flawed (so that it is either slightly lighter or heavier than the others), describe a strategy for finding the flawed coin which requires at most three weighings.

(Note : I don't remember if the strategy also reveals whether the flawed coin is light or heavy).

13. Let G be a graph and n a positive integer. Define $\chi_G(n)$ to be the number of ways to vertex-color G using at most n colors. More precisely, $\chi_G(n)$ is the number of (not necessarily surjective !) functions

$$f : V(G) \rightarrow \{1, ..., n\}$$

such that $f(x) \neq f(y)$ whenever vertices x and y are adjacent in G.

Your task is to find formulas for $\chi_G(n)$ for the following graphs :

(a) G is a tree on k vertices.
(b) G is the complete graph on k vertices.
(c) G is the cycle of length k.
(d) G is the graph to the left in Fig. 8.5.

(Note : In parts **(a)**, **(b)** and **(c)**, your formula will be a function of both k and n).