MAN 240 : Diskret matematik

Tentamen 040604

Lösningar

F.1 (i) Choose 1 keeper from 3, 4 defenders from 8, 4 midfielders from 8 and 2 forwards from 4. Applying MP, the answer is

$$\binom{3}{1} \cdot \binom{8}{4} \cdot \binom{8}{4} \cdot \binom{4}{2}.$$

(ii) 37 indistinguishable objects (the goals scored) are to be placed in 23 distinguishable cells (the players). The number of ways to do this is

$$\binom{37+23-1}{23-1} = \binom{59}{22}.$$

F.2 (i) The graph has Hamilton cycles, for example

$$a \to c_1 \to b \to g \to m_1 \to z \to m_2 \to h \to j \to d \to c_2 \to c_3 \to a.$$

(ii) $\chi(G^*) \geq 4$ since, for example, the vertex c_2 is surrounded by an odd cycle formed by a, c_1, b, d, c_3. On the other hand, if we apply the greedy algorithm to the nodes ordered left-to-right and top-to-bottom, then we get a 4-coloring, namely (the colors are $1, 2, 3, 4$)

a	1	g	2
c_1	2	h	3
c_2	3	j	1
c_3	2	m_1	1
b	1	m_2	2
d	4	z	3

Hence $\chi(G^*) = 4$.

(iii) Use BFS, starting, say, from the vertex a, to build up the following sequence of edges in a MST :

$$\{a, c_1\}, \{c_1, c_2\}, \{c_2, b\}, \{c_2, c_3\}, \{c_3, d\}, \{b, g\},$$
$$\{g, m_1\}, \{m_1, m_2\}, \{m_2, j\}, \{j, h\}, \{m_1, z\} \text{ (or } \{m_2, z\}).$$

The total weight of this tree is 15+15+10+15+10+15+10+10+5+10+25 = 140.

(iv) Apply Dijkstra's algorithm to build up the following tree

Step	Choice of edge	Labelling
1	ac_1	$c_1 := 15$
2	ac_2	$c_2 := 20$
3	ac_3	$c_3 := 25$
4	c_2b	$b := 30$
5	c_2d/c_3d	$d := 35$
6	bg	$g := 45$
7	bh	$h := 45$
8	dj	$j := 50$
9	gm_1	$m_1 := 55$
10	jm_2	$m_2 := 55$
11	m_1z/m_2z	$z := 80$

Hence the shortest path from a to z has length 80. Depending on the choices you made in Steps 5 and 11, there are three possibilities for the shortest path, namely

$$a \to c_2 \to b \to g \to m_1 \to z,$$
$$a \to c_2 \to d \to j \to m_2 \to z,$$
$$a \to c_3 \to d \to j \to m_2 \to z.$$

(v) Starting with the null flow $f \equiv 0$, one can find the following sequence of f-augmenting paths from a to z (these are not the only choices, of course) :

$$a - c_1 - b - h - m_1 - z, \quad \epsilon = 15,$$
$$a - c_3 - c_2 - d - h - m_2 - z, \quad \epsilon = 15,$$
$$a - c_2 - b - g - m_1 - z, \quad \epsilon = 10,$$
$$a - c_3 - d - j - m_2 - z, \quad \epsilon = 5.$$

This yields the following maximal flow with $|f| = 45$

Edge	Flow	Edge	Flow	Edge	Flow
(a, c_1)	15	(c_3, d)	5	(g, m_1)	10
(a, c_2)	10	(d, b)	0	(h, m_1)	15
(a, c_3)	20	(b, g)	10	(h, m_2)	15
(c_2, c_1)	0	(b, h)	15	(j, m_2)	5
(c_3, c_2)	15	(d, h)	15	(m_2, m_1)	0
(c_1, b)	15	(d, j)	5	(m_1, z)	25
(c_2, b)	10	(g, h)	0	(m_2, z)	20
(c_2, d)	15	(h, j)	0		

The corresponding minimal cut is $S = \{a, c_1, c_2, c_3, b, d, g, h, j\}$, $T = \{m_1, m_2, z\}$. Its' capacity is given by

$$c(S,T) = c(g, m_1) + c(h, m_1) + c(h, m_2) + c(j, m_2) = 10 + 15 + 15 + 5 = 45, \quad \text{v.s.v..}$$

F.3 Theorem 10.4 (17.4) in Biggs.

F.4 Let X denote the set of all permutations of $\{1, ..., 100\}$. So $|X| = 100!$. To simplify notation, put $a := \pi(1)$, $b := \pi(2)$, $c := \pi(3)$. We consider three subsets of X, namely

$$A := \{\pi \in X : a \text{ is adjacent to } b\},$$
$$B := \{\pi \in X : a \text{ is adjacent to } c\},$$
$$C := \{\pi \in X : b \text{ is adjacent to } c\}.$$

Then the number we wish to compute is simply $|X \backslash (A \cup B \cup C)|$. By the Inclusion-Exclusion principle,

$$|X \backslash (A \cup B \cup C)| = |X| - |A| - |B| - |C|$$
$$+|A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|.$$

Note that, by symmetry,

$$|A| = |B| = |C|$$

and, similarly,

$$|A \cap B| = |A \cap C| = |B \cap C|.$$

First, $|A| = 2 \cdot 99!$. For there are two choices for the ordering of a and b (either ab or ba). Then we consider these as a single 'number', which leaves

us with 99 'numbers' to permute freely.
Second, $|A \cap B| = 2 \cdot 98!$. For there are two possibilities for the ordering of a, b and c (either bac or cab). Then we consider these as a single 'number', which leaves us with 98 'numbers' to permute freely.
Third, $|A \cap B \cap C| = 0$, since it is not possible to position the three letters a, b, c so that each is adjacent to every other.

Putting it all together, we get our answer, namely

$$100! - 6 \cdot 99! + 6 \cdot 98!$$

F.5 **(i)** Theorem 5.1 (12.1) in Biggs.
(ii) See my extra lecture notes for Day 7.
(iii) See my extra lecture notes for Day 3.

F.6 Let

$$F(x) = \sum_{n=0}^{\infty} u_n x^n$$

denote the generating function of the sequence (u_n). Let's rock !

$$\begin{aligned}(2 - 7x + 3x^2)F(x) &= 2(u_0 + u_1 x) - 7(u_0 x) + \sum_{n=2}^{\infty}(2u_n - 7u_{n-1} + 3u_{n-2})x^n \\ &= 2(1 + x) - 7x + \sum_{n=2}^{\infty} 3^n x^n \\ &= 2 - 5x + \frac{9x^2}{1 - 3x} \\ &= \frac{24x^2 - 11x + 2}{1 - 3x}.\end{aligned}$$

Since

$$2 - 7x + 3x^2 = (2 - x)(1 - 3x),$$

we conclude that

$$F(x) = \frac{24x^2 - 11x + 2}{(2 - x)(1 - 3x)^2}.$$

We seek a partial fraction decomposition

$$\frac{24x^2 - 11x + 2}{(2-x)(1-3x)^2} = \frac{A}{2-x} + \frac{B}{1-3x} + \frac{C}{(1-3x)^2}. \tag{1}$$

Clearing denominators, we have

$$24x^2 - 11x + 2 = A(1-3x)^2 + B(2-x)(1-3x) + C(2-x).$$

Gathering coefficients, we get the following system of linear equations to solve

$$\begin{pmatrix} 1 & 2 & 2 \\ 6 & 7 & 1 \\ 9 & 3 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \\ 24 \end{pmatrix}.$$

After the usual Gauβ elimination and back substitution (I omit the details), we get the solution

$$A = \frac{76}{25}, \quad B = -\frac{28}{25}, \quad C = \frac{3}{5}.$$

Substituting into (1) and using the relations

$$\begin{aligned} \frac{1}{1-t} &= \sum_{n=0}^{\infty} t^n, \\ \frac{1}{(1-t)^2} &= \sum_{n=0}^{\infty} (n+1)t^n, \end{aligned}$$

we conclude that

$$F(x) = \frac{38}{25} \sum_{n=0}^{\infty} (x/2)^n - \frac{28}{25} \sum_{n=0}^{\infty} (3x)^n + \frac{3}{5} \sum_{n=0}^{\infty} (3x)^n.$$

It follows that

$$u_n = \frac{38}{25} \cdot \frac{1}{2^n} - \frac{28}{25} \cdot 3^n + \frac{3}{5} \cdot (n+1) \cdot 3^n.$$