MATEMATIK Göteborgs Universitet Peter Hegarty

Dag : 050117 Tid : 8.30 - 13.30.
Hjälpmedel : Inga
Vakt : Johanna Pejlare 076-2186654.

Tentamenskriving i Talteori (MAN 640)

$\geq$ 12.5 poäng, inkl. inlämningsuppgifterna, ger godkännt.

1 (3p). Determine all primitive roots modulo 31.

2 (3p). Establish, with proof, the connection between Mersenne primes and perfect numbers.

3 (3p). Let p be a prime congruent to 3 modulo 4. Let m be the number of quadratic non-residues in the interval $[1, p/2)$. Prove that

$$\left[\frac{1}{2}(p-1)\right]! \equiv (-1)^m \pmod{p}.$$

4 (4p). With the help of the identity (which you don't need to motivate)

$$\begin{gathered}(x^2+y^2+z^2+w^2)(a^2+b^2+c^2+d^2) = \\ (xa+yb+zc+wd)^2+(xb-ya+wc-zd)^2 \\ +(xc-za+yd-wb)^2+(xd-wa+zb-yc)^2,\end{gathered}$$

prove Lagrange's theorem that every positive integer is a sum of 4 integer squares.

5 (3p). Let (x, y, z) be a Pythagorean triple. Prove that xyz is divisible by 60.

6 (0.5p+2.5p) (i) State Dirichlet's approximation theorem.
(ii) Using this result (or otherwise), prove that, if $d > 0$ is not a perfect square, then the equation $x^2 - dy^2 = 1$ has a non-trivial integer solution.

7 (2p+2p) (i) Determine (with proof) all reduced binary quadratic forms

of discriminant -27.
(ii) Give a variable substitution which converts the form

$$103x^2 + 73xy + 13y^2$$

to a reduced form (OBS! the form has discriminant -27).

8 (0.5p+2.5p) (i) Write down (no proof needed !) the Euler product for $\zeta(s)$ and state the range of $s \in \mathbf{C}$ in which the representation is valid.
(ii) Hence, or otherwise, prove that the sum of the reciprocals of the primes diverges.

Obs! Tentan beräknas vara färdigrättad den 24 januari. Då kan den hämtas i mottagningsrummet mellan kl. 12:30-13:00. Tentamensresultat lämnas också ut per telefon 772 35 09 *efter* kl. 14:00.