MAN 640 : Taltaori

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F.1 1003 factorises as 17×59 so, by the Chinese Remainder Theorem, there's a solution to the congruence modulo 1003 if and only if there is a solution modulo both 17 and 59. In general, there is a solution mod p if and only if the discriminant of the quadratic, $b^2 - 4ac = 11^2 - 4 \cdot 3 \cdot 9 = 13$ is a quadratic residue mod p. So we need to know whether 13 is a quadratic residue modulo 17 and 59. Using quadratic reciprocity we compute that

$$\left(\frac{13}{17}\right) = \left(\frac{17}{13}\right) = \left(\frac{4}{13}\right) = \left(\frac{2}{13}\right)^2 = 1,$$

$$\left(\frac{13}{59}\right) = \left(\frac{59}{13}\right) = \left(\frac{7}{13}\right) = \left(\frac{13}{7}\right) = \left(\frac{-1}{7}\right) = -1.$$

Thus the congruence has no solution.

F.2 Theorem 6 in my lecture notes from 2004.

F.3 It's either rational or a quadratic irrational. This is a classical theorem of Lagrange. See section 7.7 of NZM for an explanation.

F.4 One proof is Theorem 25 of my 2004 lecture notes. Consult the handouts from the book of Stewart and Tall for the proof using Minkowski's theorem.

F.4 (i) Let $\{a, b, c\}$ be a reduced positive-definite form of discriminant -44. Since $b^2 - 4ac = -44$ is even, we must have b even. Since the form is reduced we have

$$0 < a \leq \sqrt{\frac{-d}{3}} \Rightarrow a \in \{1, 2, 3\}.$$

If $a = 1$ then, since $b \in (-a, a]$, the only possibility is $b = 0$. This gives $c = 11$, so we have the form $\{1, 0, 11\}$.

If $a = 2$ then $b \in \{0, 2\}$, in which case $c = (b^2 + 44)/8$ is an integer when $b = 2$. We get the form $\{2, 2, 6\}$.

If $a = 3$ then $b \in \{0, \pm 2\}$. We'll get an integer-valued c if $b = \pm 2$, thus giving us two further reduced forms, namely $\{3, \pm 2, 4\}$.

We conclude that there are four reduced forms of discriminant -44, namely

$$x^2 + 11y^2, \quad 2x^2 + 2xy + 6y^2 \quad \text{and} \quad 3x^2 \pm 2xy + 4y^2.$$

(ii) Denote the given form as $f(x, y) = \{113, 42, 4\}$. We apply the following sequence of transformations to reduce the form :

$$\begin{aligned} S &: \{113, 42, 4\} \mapsto \{4, -42, 113\}, \\ T^5 &: \{4, -42, 113\} \mapsto \{4, -2, 3\}, \\ S &: \{4, -2, 3\} \mapsto \{3, 2, 4\}. \end{aligned}$$

Hence f is equivalent to the reduced form $3x^2 + 2xy + 4y^2$. To work out the variable substitution which accomplishes this transformation, we compute

$$\begin{aligned} ST^5S = (ST^5)S &= \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 5 & -1 \end{pmatrix}. \end{aligned}$$

Hence the desired variable substitution is

$$f(-x, 5x - y) = 3x^2 + 2xy + 4y^2.$$

F.6 See lecture notes.

F.7 **(i)** For $\text{Re}(s) > 1$, the following representation is valid :

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

See Proposition 21 in the 2004 lecture notes for an outline of the proof.
(ii) Theorem 27 in the 2004 lecture notes.

F.8 (i) $W(k,m)$ is the smallest positive integer n for which any m-coloring of the set $\{1,...,n\}$ must yield a monochromatic k-term AP.
(ii) We consider a random m-coloring of $\{1,...,n\}$ and show that, if

$$n < \sqrt{2(k-1)}m^{\frac{k-1}{2}} \qquad (1)$$

then there is a positive probability that there is no monochromatic k-AP. The probability of any k-AP being monochromatic is $\left(\frac{1}{m}\right)^{k-1}$. The number of k-AP:s in $\{1,...,n\}$ can be estimated as follows : any k-AP is determined by its first term and common difference. If the first term is x, then the common difference can be no more than $\frac{n-x}{k-1}$. Thus the number of k-AP:s is at most

$$\frac{1}{k-1}\sum_{x=1}^{n} x(n-x).$$

We could evaluate the sum exactly, but for our purposes it is enough to check that the sum is at most $\frac{1}{2}n^2$. Then the number of k-AP:s is at most

$$\frac{n^2}{2(k-1)}.$$

By a simple union bound, it follows that the probability of their being some monochromatic k-AP is at most

$$\frac{n^2}{2(k-1)m^{k-1}}.$$

We want this probability to be strictly less than one, and this is obviously the case if and only if (1) holds. Q.E.D.