MATEMATIK
Göteborgs Universitet
Peter Hegarty

Dag : 130826   Tid : 8.30 - 13.00 (**Obs! 4.5 hours**).
Hjälpmedel : Inga
Vakter : Christoffer Standar 0703-088304,
Peter Hegarty 0766-377873.

Tentamenskrivning i Talteori (MMA 300)

≥ 50 points, including bonuses from the homeworks, required to pass. In Problems 1,3,5,7, any results that you use from the lecture notes may be just stated without proof.

1 (8p) Determine with proof all primitive roots modulo 61.

2 (7p+8p) (i) Prove that there are infinitely many primes congruent to 1 (mod 4).
(i) Prove that the sum of the reciprocals of the primes diverges.

3 (5p+7p) (i) Prove that, for any natural number n,

$$\sum_{d|n} \phi(d) = n.$$

(ii) Prove or disprove the following statement: *The set* $S = \{\phi(n)/n : n \in \mathbb{N}\}$ *is dense in the closed interval* $[0, 1]$.

4 (14p) State and prove Lagrange's theorem on sums of four squares.

5 (10p) Prove that if $A \subseteq \{1, ..., n\}$ has size greater than $n/2$, then $A + A$ contains an arithmetic progression of length $\Omega(n)$, i.e.: of length at least $c \cdot n$, where $c > 0$ is an absolute constant.

6 (14p) State and prove the Cauchy-Davenport-Chowla theorem for sumsets in $\mathbb{Z}_p$, where p is a prime.

7 (2p+8p) (i) Define the Van der Waerden number $W(k,l)$.
(ii) Using a probabilistic method, or otherwise, show that

$$W(k,l) > \sqrt{2(k-1)}l^{(k-1)/2}.$$

8 (2p+15p) (i) State the Regularity Lemma.
(ii) Using the Regularity Lemma, give a complete proof of Roth's theorem, i.e.: of the fact that, if $f(n)$ is the maximum size of a 3-AP-free subset of $\{1, ..., n\}$, then $f(n) = o(n)$.

Obs! Tentan beräknas vara färdigrättad den 2 september. Då kan den hämtas i mottagningsrummet mellan kl. 12:30-13:00. Tentamensresultat lämnas också ut per telefon 772 35 09 *efter* kl. 14:00.