THEOREM 1. *Let A be a subset of $[1, n]$ containing no solutions to the equation*

$$3x = y + z. \tag{6}$$

Then either $n = 4$ and $A = \{1, 3, 4\}$ or $|A| \leq \lceil \frac{n}{2} \rceil$.

REMARK. For every $n \geq 1$ the set A of odd integers in $[1, n]$ has size $\lceil \frac{n}{2} \rceil$ and contains no solutions to (6).

Proof of Theorem 1. One may verify by hand that the result holds for all $n < 10$. Now we proceed by induction on n. So let $n \geq 10$ and assume the result holds for all $i < n$. The induction step is trivial if n is odd, so we may assume that n is even. Let $s = \lfloor \frac{n}{3} \rfloor$ and $t = \lfloor \frac{2n}{3} \rfloor$. Partition the interval $[1, n]$ into three subintervals

$$I_1 = [1, s], \quad I_2 = (s.t], \quad I_3 = (t, n].$$

Let A be a subset of $[1, n]$ avoiding (6). Suppose $A \cap I_2 \neq \phi$ and let $x \in A \cap I_2$. Then $1 \leq 3x - n \leq n$. Let $I_x := [3x - n, n]$. The map

$$f : y \mapsto 3x - y$$

is a bijection from I_x to itself. If, for some $y \in I_x$, both y and $f(y)$ lay in A, then we'd have a solution in A to (6), namely

$$3x = (3x - y) + y.$$

It follows that $|A \cap I_x| \leq \lfloor \frac{1}{2} |I_x| \rfloor$. But the induction hypothesis gives an upper bound on $|A \cap [1, 3x - n)|$ and tells us that the only possible way to achieve $|A| > \lceil \frac{n}{2} \rceil$ is if the following hold :

(i) $A \cap I_2 = \{x\}$ where $3x - n = 5 \Leftrightarrow x = \frac{n+5}{3}$,
(ii) $\{1, 3, 4\} \subset A$,
(iii) $|A \cap I_x| = \frac{1}{2} |I_x|$.

Thus we have two cases to consider.

CASE I : $A \cap I_2 = \phi$. Since the induction hypothesis gives an upper bound on $|A \cap I_1|$ it's easy to see that it only allows for the possibility that $|A| > \lceil \frac{n}{2} \rceil$ if $I_3 \subset A$. Then we have three cases left :

(Φ_1) $n \equiv 0 \pmod 3$, so $n = 3l$ with l even. $I_1 = [1, l]$, $I_2 = [l+1, 2l]$ and $I_3 = [2l+1, 3l]$. So, by induction, $|A| > \lceil \frac{n}{2} \rceil$ is only possible if $l = 4$, $n = 12$ and

$$A = \{1, 3, 4, 9, 10, 11, 12\}.$$

But then $3 \cdot 4 = 1 + 11$ is a solution in (6) in A, contradiction.

(Φ_2) $n \equiv 2 \pmod 3$, so $n = 3l+2$ with l even. $I_1 = [1, l]$, $I_2 = [l+1, 2l+1]$ and $I_3 = [2l+2, 3l+2]$. By induction, $|A| > \lceil \frac{n}{2} \rceil$ is only possible if $l = 4$, $n = 14$ and

$$A = \{1, 3, 4, 10, 11, 12, 13, 14\}.$$

But then $3 \cdot 4 = 1 + 11$ is once again a solution to (6) in A, a contradiction.

(Φ_3) $n \equiv 1 \pmod 3$, so $n = 3l + 1$ with l odd. $I_1 = [1, l]$, $I_2 = [l+1, 2l]$ and $I_3 = [2l+1, 3l+1]$. By induction, if $|A| > \lceil \frac{n}{2} \rceil$, then in addition to $I_3 \subset A$ we must have that $|I_1 \cap A| = \frac{l+1}{2}$. With regard to the latter, induction again implies that either

(a) $l \in A$, or

(b) $l = 5$ and $I_1 \cap A = \{1, 3, 4\}$.

If (b) holds, then $n = 16$ and $A = \{1, 3, 4, 11, 12, 13, 14, 15, 16\}$. But then $3 \cdot 4 = 1 + 11$ is yet again a solution to (6) in A, contradiction.

If (a) holds then since $n \geq 10$ we have $l \geq 3$, so $I_1 \cap A$ contains at least one further element $m < l$. But then $3l - m$ cannot lie in A as otherwise we'd have the solution

$$3 \cdot l = (3l - m) + m$$

to (6). But $3l - m \in I_3$, contradicting the fact that $I_3 \subset A$.

This completes the induction step in Case I.

Case II : The conditions (i), (ii) and (iii) above are satisfied. By (i), $\frac{n+5}{3}$ is an integer, so $n = 3l + 1$ for some odd l. Then $I_1 = [1, l]$, $I_2 = [l+1, 2l]$, $I_3 = [2l+1, 3l+1]$ and $A \cap I_2 = \{l + 2 = x\}$. Since $n \geq 10$ we have that $2x \in I_3$. But $2x \notin A$, as otherwise we'd have the solution

$$3 \cdot x = (2x) + x$$

to (6) in A. By induction, we conclude that the only way $|A| > \lceil \frac{n}{2} \rceil$ is possible is if $I_3 \backslash A = \{2x\}$ and $|I_1 \cap A| = \frac{l+1}{2}$. As in Case I above, we can further conclude that either

(a) $l \in A$, or

(b) $l = 5$ and $I_1 \cap A = \{1, 3, 4\}$.

If (b) holds then $n = 16$ and $A = \{1, 3, 4, 7, 11, 12, 13, 15, 16\}$, so once again we have the contradiction that $3 \cdot 4 = 1 + 11$ is a solution to (6) in A.

If (a) holds and $l > 3$, then there are at least two distinct numbers $m_1 < m_2 < l$ in $I_1 \cap A$. Arguing as in Case I, we obtain the contradiction that $I_3 \backslash A$ contains at least two elements. Finally, then, we are left with $l = 3$, in which case $n = 10$ and $x = 5$. But then $2x = 10 \notin A$, so $\{5, 7, 8\} \subset A$ and $3 \cdot 5 = 7 + 8$ is a solution to (6) in A.

This final contradiction completes the induction step in Case II, and with it the proof of Theorem 1.

COROLLARY.

$$\lambda_{0,3} = \lambda_{1,3} = \lambda_{2,3} = \lambda_{3,3} = \lambda_{4,3} = \rho_3 = \frac{1}{2}.$$

Proof. Immediate.

3. The case $k \geq 4$. The main result is Theorem 3, which is a general result valid for all $k \geq 4$. We fear, however, that if the reader were to study that proof immediately, then he/she may drown in the algebra and not see the main ideas so clearly. We have therefore decided to first present, in Theorem 2 below, the complete proof in the special case $k = 4$.

First, some simplifying terminology :

DEFINITION 1. A set of positive integers which contains no solutions to the equation

$$4x = y + z$$

will be called *good*.