Homework 3 (due Wednesday, Jan. 7)

There is a total of 30 points for exercises without a *. The exercises with a * are considered a lot more difficult. Bonus points are calculated as $x/5$, where the maximum possible value of x is 42. All your work must be properly motivated !

In Exercises 1-2 you will require the following terminology : Let $\mathcal{L}$: $\sum_{i=1}^{n} a_i x_i = a_0$ be a linear Diophantine equation. Let $c \in \mathbb{N}$. We say that the equation $\mathcal{L}$ is *c-irregular* if there exists a c-coloring of $\mathbb{N}$ for which there are no non-trivial monochromatic solutions to $\mathcal{L}$. We say that $\mathcal{L}$ is *irregular* if it is c-irregular for some $c \in \mathbb{N}$. Otherwise, $\mathcal{L}$ is said to be *(partition) regular*.

Q.1 (3p) Let $\mathcal{L}$ be a linear Diophantine equation. Suppose that there exists a $c \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$, there exists a c-coloring of $\{1, ..., n\}$ which induces no non-trivial monochromatic solutions to $\mathcal{L}$. Prove that $\mathcal{L}$ is then irregular.

Q.2 (3p) Prove that the equation $4x = 2y + z$ is 3-irregular.

(REMARK : The question of whether one can 3-color the reals such that there are no monochromatic solutions to this equation is known to be undecidable in ZFC-set theory).

Q.3 (2p) Let $h \geq 2$. Recall that an asymptotic basis A for $\mathbb{N}_0$ of order h is said to be *thin* if the counting function $A(n)/n^{1/h}$ is bounded. Let's call A *skinny* if the representation function $r_{A,h}(n)$ is bounded.

Prove that a skinny asymptotic basis of order h is also thin of order h. On the other hand, give an example for each $h \geq 2$ of a thin asymptotic basis of order h which is not skinny.

Q.4 (2p) Without using generating functions, show that it is impossible for a subset $A \subseteq \mathbb{N}$ to satisfy $r_2(A, n) = 1$ for all $n \gg 0$.

Q.5 (2p) Prove that the Erdős-Turán Conjecture fails in $\mathbb{Z}$ by exhibiting, for each $h \geq 1$, a basis A for $\mathbb{Z}$ of order h such that $r_{A,h}(n) = 1 \; \forall \; n \in \mathbb{Z}$.

Q.6 (4p+1p+1p+4p) Let A be an asymptotic basis for $\mathbb{N}_0$. An element $a \in A$ is said to be *essential* if the set $A \backslash \{a\}$ is no longer an asymptotic basis, of any order.

***(i)** Prove that if A is an asymptotic basis for $\mathbb{N}_0$ and $a \in A$, then a is essential if and only if $A \backslash \{a\}$ is contained inside some non-trivial, homogeneous arithmetic progression, i.e.: inside $n\mathbb{Z}$ for some $n > 1$.
(ii) Deduce from part (i) that an asymptotic basis contains only finitely many essential elements.
(iii) For each $k \geq 1$, give an example of an asymptotic basis with exactly k essential elements.
***(iv)** Prove that there exists a function $X : \mathbb{N} \to \mathbb{N}$ with the following property :

For every $h \in \mathbb{N}$, every asymptotic basis A for $\mathbb{N}_0$ of order h and every $a \in A$, either a is essential or the order of $A \backslash \{a\}$ as an asymptotic basis is at most $X(h)$. In fact, prove that, as $h \to \infty$,

$$\frac{h^2}{4} \lesssim X(h) \lesssim \frac{5h^2}{4}.$$

Q.7 (5x1p + 4p) If $A \subseteq \mathbb{Z}$, then the *difference set* $A - A$ is defined as

$$A - A = \{a_1 - a_2 : a_1, a_2 \in A\}$$

and the *restricted sumset* is defined as

$$A \hat{+} A = \{a_1 + a_2 : a_1,\ a_2 \in A,\ a_1 \neq a_2\}.$$

(i) For a finite set A, give upper and lower bounds for $|A - A|$ in terms of $|A|$, analogous to those given in the lectures for the sumset $A + A$.
(ii) Give any example whatsoever of a finite set A for which $|A + A| > |A - A|$.
(iii) A set A is said to be *symmetric* if there exists $x \in \mathbb{Z}$ such that $A = \{x\} - A$. Prove that, if A is symmetric, then $|A + A| = |A - A|$.
(iv) Prove that $|A \hat{+} A| \leq |A + A| - 2$.
(v) Let A be a symmetric set and $x \in \mathbb{Z} \backslash A$. Let $B := A \cup \{x\}$. Prove that $|B \hat{+} B| < |B - B|$.
***(vi)** Prove that there exists a real number $C > 0$ such that, for all $n \in \mathbb{N}$, there are at least $C \cdot 2^n$ subsets A of $\{1, ..., n\}$ which satisfy $|A + A| \geq |A - A|$.

Q.8 (2p) For each $n \in \mathbb{N}$, let

$$A_n := \{k^2 : 1 \leq k \leq n\}.$$

Prove that, for any $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{|A_n - A_n|}{n^{2-\epsilon}} = +\infty.$$

Q.9 (2p+3p) Let $A \subseteq \mathbb{Z}$ be a finite set, $|A| = k$.
(i) Prove that, if $|A + A| = 2k - 1$, then A is an arithmetic progression.

(ii) Prove that, if $k > 3$ and $|A + A| \leq 2k$, then there exists an arithmetic progression B such that $|B| \leq k + 1$ and $A \subseteq B$).

Q.10 (2p) Prove or disprove the existence of a countably infinite subset $A \subset [0, 1]$ which has distinct subset sums.

Q.11 (3p) Complete the proof of Chernoff's inequality by proving (in the notation of the lecture notes) that

$$\mathbb{P}(\hat{X} < -a) \leq \exp\left(\frac{-a^2}{2pn}\right).$$