MATEMATIK
Göteborgs Universitet
Peter Hegarty

Dag : 150116   Tid : 8.30 - 13.00 (**Obs! 4.5 hours**).
Hjälpmedel : Inga
Vakter : Åse Fahlander 0703-088304,
Peter Hegarty 0766-377873.

Tentamenskriving i Talteori (MMA 300)

≥ 50 points, including bonuses from the homeworks, required to pass. In Problems 1,3,5,7, any results that you use from the lecture notes may be just stated without proof.

1 (5p+5p) (i) Find, with proof, any primitive root a modulo 67, and list all the primitive roots in the form a^k, where $k \in \mathbb{N}$.
(ii) Is 29 a square modulo 67 ? Motivate your answer, of course.

2 (15p) Determine with proof all positive integers which can be expressed as the sum of two squares.

3 (10p) Let d be a positive integer. Prove that d is a perfect square if and only if the congruence $x^2 \equiv d \pmod{p}$ has a solution for every prime p.

4 (15p) Prove that

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + b + O\left(\frac{1}{\log x}\right),$$

where b is some constant.

5 (10p) Let $g_2(n)$ be the maximum size of a subset of $\{1, 2, \ldots, n\}$ in which no number is an integer multiple of any other. Prove that $g_2(n) = \lfloor \frac{n+1}{2} \rfloor$.

6 (2p+2p+11p) (i) State Chernoff's inequality as formulated in the lectures.
(ii) State the Borel-Cantelli lemma.
(iii) With the help of the above, and the fact that

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi,$$

prove that there exists an asymptotic basis A of order 2 for $\mathbb{N}$ such that $r_2(A, n) = \Theta(\log n)$.

7 (10p) Let n be a prime. For $p = p(n) \in [0, 1]$, let $A = A(n, p)$ be a random subset of $\mathbb{Z}_n$ obtained by choosing each element of $\mathbb{Z}_n$ independently with probability p. Prove that, as $n \to \infty$, if $\frac{p(n)}{n^{-2/3}} \to \infty$ then

$$\mathbb{P}(A \text{ contains no 3-term AP mod } n) \to 0.$$

(HINT: Do a second moment analysis.)

8 (3p+7p+5p) (i) State the Regularity Lemma, including full definitions of all terms used, such as ε-*regular pair* and ε-*regular partition*.
(ii) Assuming the Regularity Lemma, state and prove the Triangle Counting Lemma.
(iii) Using the above, state and prove Roth's theorem.

Obs! Tentan beräknas vara färdigrättad den 23 januari. Då kan den hämtas i expeditionen (ankn. 3500) mellan kl. 11:00-13:00, alla vardagar utom onsdagar.