

# MINIMAL ODD ORDER AUTOMORPHISM GROUPS

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ABSTRACT. We show that  $3^7$  is the smallest order of a non-trivial odd order group which occurs as the full automorphism group of a finite group.

## 1. INTRODUCTION AND NOTATION

The map which takes every element of a group to its inverse is an automorphism if and only if the group is abelian. Following [HL], we say that a group is N.I. (no inversions occur) if no automorphism of the group sends any non-identity element to its inverse<sup>1</sup>. Thus an N.I. group is non-abelian. It is easy to see that a finite group  $G$  is N.I. only if  $\text{Aut}(G)$  has odd order<sup>2</sup>, and that the order of such a group must be odd. N.I. groups have attracted some attention, because of the surprising difficulty in finding explicit examples of them. Though there are a number of results indicating that such groups are ubiquitous (for example, see [HL], [HR] and [M]), no brute force search among ‘small’ groups will yield any examples. More precisely, MacHale and Sheehy [MS] proved that there is no finite N.I. group of order less than  $3^6$ . There exist groups of order  $3^6$  with automorphism group of order  $3^7$  (see the Small Groups Library [BEB]). In this paper we prove the following complement to MacHale and Sheehy’s result :

**Theorem 1.1.** *If  $G$  is a non-trivial finite group such that  $|\text{Aut}(G)|$  is odd, then  $|\text{Aut}(G)| \geq 3^7$ .*

The proof will be presented in the next two sections. In Section 4 we present some remaining open problems concerning small N.I. groups.

All groups considered in the remainder of this paper are finite. The following notational conventions will be adhered to throughout.

We denote by  $\Omega(n)$  the number of prime factors of the positive integer  $n$ , counting repetitions, and by  $\omega(n)$  the number of distinct prime factors of  $n$ . If  $G$  is a group then  $\pi(G)$  denotes the set of distinct prime factors of  $|G|$ . If  $p \in \pi(G)$  then  $G_p$  denotes a Sylow  $p$ -subgroup of  $G$ .

The direct product of  $n$  copies of a group  $G$  with itself will be denoted  $G^n$ .

The center of a group  $G$  is denoted by  $Z(G)$ . We will often just write  $Z$  when it is clear to which group we are referring.

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<sup>1</sup>In [MS], the term I.F.P. (inverse-point-free) is used instead.

<sup>2</sup>We do not know if there exists an infinite group which has no automorphism of order 2 but is not N.I.

$\text{Inn}(G)$  and  $\text{Cent}(G)$  denote, respectively, the groups of inner and central automorphisms of  $G$ . For  $g \in G$ , the inner automorphism  $x \mapsto g^{-1}xg$  will be denoted  $I_g$ . The commutator  $x^{-1}g^{-1}xg$  will be denoted  $[x, g]$ .

Let  $A$  be a subgroup of  $G$ . Let  $\phi \in \text{Aut}(A)$  and  $\phi^* \in \text{Aut}(G)$ . We say that  $\phi^*$  is an *extension* of  $\phi$  (or, equivalently, that  $\phi$  *extends* to  $\phi^*$ ), if  $(A)\phi^* = A$  and  $\phi^*|_A = \phi$ .

If  $A$  is a normal subgroup of  $G$ , we say that  $G$  *splits* over  $A$  if there is a subgroup  $B \subseteq G$  such that  $A \cap B = \{1\}$  and  $G = AB$ . Then  $G$  is said to be a *split extension* of  $A$  by  $B$  and we write  $G = A \rtimes B$ . The subgroup  $B$  is said to be a *complement* of  $A$ . It acts on  $A$  by conjugation and the induced subgroup of  $\text{Aut}(A)$  is denoted by  $A^B$ . This last piece of notation will be used even in the case of a non-split extension.

If  $q$  is a prime power, then  $\mathbb{F}_q$  denotes the finite field with  $q$  elements and  $\text{GL}(n, q)$  the group of invertible linear transformations on the  $n$ -dimensional vector space over  $\mathbb{F}_q$ . If  $q$  is a prime, this group can be identified with  $\text{Aut}(G)$  where  $G \cong C_p^n$ , the elementary abelian  $p$ -group of rank  $n$ .

## 2. TOOLBOX

All of the results in this section, except for Theorem 2.13, are taken from the existing literature. The first seven lemmas are, moreover, standard material. For these the reader seeking details beyond what we provide is referred to [Sco] in the absence of any alternative citations.

**Lemma 2.1.**  $\text{Cent}(G) \cap \text{Inn}(G) = Z[\text{Inn}(G)] \cong Z[G/Z(G)]$ .

**Lemma 2.2.** *Let  $G$  be a group without any abelian direct factor. Then*

$$|\text{Cent}(G)| = |\text{Hom}(G/G', Z(G))|. \quad (2.1)$$

*Proof.* The natural one-to-one correspondence is given by  $\phi \rightarrow (G'g \mapsto g^{-1}g\phi)$ . For a rigorous proof of the lemma, see [Sa].  $\square$

**Lemma 2.3.** *Let  $G$  be a group with center  $Z$  and suppose  $G/Z$  is a  $p$ -group for some prime  $p$ .*

(i) *If  $G/Z$  is abelian, say*

$$G/Z \cong \prod_{i=1}^r C_{p^{\alpha_i}}, \quad (2.2)$$

*with  $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_r$ , then  $r \geq 2$  and  $\alpha_1 = \alpha_2$ .*

(ii) *If  $G/Z$  is not elementary abelian then it possesses a non-trivial central characteristic subgroup  $K$  of index at least  $p^2$ .*

*Proof.* (i) Clearly  $r \geq 2$ . Let  $Za$  be a generator of a highest order cyclic factor. Then for any  $Zb$  in any complementary factor we have  $b^{p^{\alpha_2}} \in Z$ . Since  $G/Z$  is abelian we have  $1 = [a, b^{p^{\alpha_2}}] = [a, b]^{p^{\alpha_2}} = [a^{p^{\alpha_2}}, b]$ . Thus  $a^{p^{\alpha_2}} \in Z$ , so  $\alpha_1 = \alpha_2$ .

(ii) If  $G/Z$  is not abelian, take  $K$  to be its center. Otherwise take  $K$  to be the subgroup generated by the elements of order  $p$ . In both cases  $K$  is non-trivial of index at least  $p^2$ , by part (i).  $\square$

**Lemma 2.4.** *Let  $G$  be a group,  $N$  a normal subgroup of  $G$ . Let  $\phi$  be an automorphism of  $G$  whose order is relatively prime to that of  $G$  and which acts trivially on both  $N$  and  $G/N$ . Then  $\phi$  is the trivial automorphism.*

*Proof.* Let  $g \in G$ . Since  $\phi$  acts trivially on  $G/N$ , we have  $g\phi = ng$  for some  $n \in N$ . Since  $\phi$  acts trivially on  $N$  it follows that for every  $i$ ,  $(g)\phi^i = n^i g$ . In particular, taking  $i = |\phi|$  we find that  $n^{|\phi|} = 1$ . But  $|\phi|$  is prime to  $|G|$  so  $n = 1$ , thus  $g\phi = g$  as desired.  $\square$

**Lemma 2.5.** *Let  $G$  be a group,  $p \in \pi(G)$  and  $P$  a Sylow  $p$ -subgroup of  $G$ .*  
 (i) *The number of conjugates of  $P$  in  $G$  is congruent to one modulo  $p$ .*  
 (ii) *If  $P \triangleleft G$  then  $G$  splits over  $P$ .*

**Lemma 2.6.** *For any positive integer  $n$  and prime power  $q$ , we have*

$$|GL(n, q)| = \prod_{i=0}^{n-1} (q^n - q^i). \quad (2.3)$$

**Lemma 2.7.** *Let  $G$  be a group of odd order. Then  $G$  possesses an elementary abelian characteristic  $p$ -subgroup, for some  $p \in \pi(G)$ .*

*Proof.* A group is characteristically simple if and only if it is the internal direct product of isomorphic simple subgroups. Since every group of odd order is soluble, it follows that a characteristically simple group of odd order must be an elementary abelian  $p$ -group. This and the fact that

$$B \text{ char } A \text{ char } G \Rightarrow B \text{ char } G, \quad (2.4)$$

allows us to prove the lemma by induction on the group order.  $\square$

**Lemma 2.8.** *If  $\Omega(|\text{Aut}(G)|) \leq 4$  then  $|\text{Aut}(G)|$  is even.*

*Proof.* This follows from the work of various authors. See the introduction to [SZ], where all the necessary references are given.  $\square$

**Lemma 2.9.** *Let  $p$  be an odd prime and  $G$  a  $p$ -group such that  $\text{Aut}(G)$  is also a  $p$ -group. Then  $|\text{Aut}(G)| \geq 3^7$  if  $p = 3$  and  $|\text{Aut}(G)| \geq p^6$  if  $p > 3$ .*

*Proof.* See [C1] and [MS].  $\square$

**Corollary 2.10.** *If  $G$  is a group for which  $\text{Aut}(G)$  is an odd order group of order strictly less than  $3^7$  then  $\text{Aut}(G)$  is not a  $p$ -group.*

*Proof.* If  $\text{Aut}(G)$  is a  $p$ -group then so is the central factor group  $G/Z$ , being isomorphic to a normal subgroup of the former. Thus  $G$  is nilpotent, hence the internal direct product of its Sylow subgroups, say  $G = P_1 \times \cdots \times P_k$ . But then also

$$\text{Aut}(G) = \prod_{i=1}^k \text{Aut}(P_i). \quad (2.5)$$

Since the automorphism group of a 2-group is either trivial or of even order, we now obtain the desired conclusion from Lemma 2.9.  $\square$

The next three results will be the main tools in our approach to proving Theorem 1.1. The first two already appear in several papers, so we omit proofs. See, for example, Theorem 2.1 of [C2].

**Lemma 2.11.** *Suppose  $G = AB$  for some pair of subgroups  $A$  and  $B$ , where  $A \triangleleft G$ . Let  $\phi \in \text{Aut}(A)$ . Then  $\phi$  extends to an automorphism  $\phi^*$  of  $G$  such that  $|\phi| = |\phi^*|$  if the following two conditions are satisfied :*

- (i)  $\phi$  is trivial on  $A \cap B$ ,
- (ii) in the group  $\text{Aut}(A)$ ,  $[\phi, A^B] = 1$ .

This lemma will be used in the case when  $\phi$  has order 2 and the subgroup  $A$  is either abelian or a  $p$ -group of class 2. In the former case we have

**Corollary 2.12.** *Suppose  $G = A \rtimes B$  where  $A$  is abelian of order greater than 2. Then  $G$  possesses an automorphism of order 2. In particular, if  $G$  has a normal, abelian Sylow  $p$ -subgroup, for some odd prime  $p$ , then  $|\text{Aut}(G)|$  is even.*

In the case of non-abelian  $A$  we have the following result. To the best of our knowledge it is a new result, and we thus state it as a theorem in its own right.

**Theorem 2.13.** *Let  $G = AB$ , where  $A$  is a normal  $p$ -subgroup of class 2 with  $A/(Z \cap A)$  elementary abelian,  $B/(Z \cap B)$  is an abelian  $p'$ -group and  $A \cap B \subseteq Z$ . Assume  $B$  acts non-trivially on  $A$ . Then  $G$  possesses an automorphism of order 2 .*

While our proof is lengthy, it is crucial to our analysis in Section 3.

*Proof.* Since  $B$  acts non-trivially on  $A$ , it does so even on  $A/(Z \cap A)$ , by Lemma 2.4. Let  $n$  be the rank of  $A/(Z \cap A)$  as an elementary abelian  $p$ -group. The group  $A^B$  thus describes a non-trivial representation of the abelian  $p'$ -subgroup  $B/(Z \cap B)$  in  $\text{GL}(n, p)$ . By Maschke's theorem, this representation is completely reducible, say

$$A^B = \psi_1 \times \cdots \times \psi_r, \quad (2.6)$$

where  $\text{rank}(\psi_i) = n_i$  say. Moreover, Schur's lemma implies that  $A^B/\ker\psi_i$  is cyclic for each  $i$ . We first prove the theorem in two special cases, before establishing the general result.

CASE 1 : Suppose that  $A^B$  is irreducible.

Let  $A^B = \langle I_b \rangle$ . Clearly, there is a choice of basis  $Z\alpha_1, \dots, Z\alpha_n$  for  $A/(Z \cap A)$ , an element  $z \in Z$  and non-negative integers  $k_1, \dots, k_n$  such that the action of  $b$  on  $A$  is given by

$$b^{-1}\alpha_i b = \alpha_{i+1}, \quad i = 1, \dots, n-1, \quad (2.7)$$

$$b^{-1}\alpha_n b = z\alpha_1^{k_1} \cdots \alpha_n^{k_n}. \quad (2.8)$$

In other words, the matrix of  $I_b$  with respect to this basis is

$$M = M_b = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ k_1 & k_2 & \cdot & \cdot & \cdot & \cdot & k_n \end{pmatrix}. \quad (2.9)$$

Let  $\mathbb{I}_n$  denote the  $n \times n$  identity matrix. One checks easily that  $\det(M - \mathbb{I}_n) = (-1)^{n+1}[(k_1 + k_2 + \cdots + k_n) - 1]$ , hence, since  $M$  acts non-trivially and irreducibly, we have

$$k_1 + k_2 + \cdots + k_n \not\equiv 1 \pmod{p}. \quad (2.10)$$

Raising both sides of (2.7) to the  $p$ -th power for each  $i$  yields

$$\alpha_1^p = \alpha_2^p = \cdots = \alpha_n^p. \quad (2.11)$$

Doing the same for (2.8) and inserting (2.11) yields

$$\alpha_n^p = z^p \alpha_n^{(k_1 + \cdots + k_n)p}. \quad (2.12)$$

By (2.10) there is a substitution  $a_n := \sigma_n \alpha_n$ , for some  $\sigma_n \in Z$ , such that  $a_n^p = 1$ . Working back through equations (2.7) one sees that one can make a sequence of similar substitutions  $a_i := \sigma_i \alpha_i$ ,  $i = n-1, n-2, \dots, 1$ , such that

$$a_1^p = a_2^p = \cdots = a_n^p = 1. \quad (2.13)$$

With respect to the elements  $a_1, \dots, a_n$ , there exist  $z_1, \dots, z_n \in Z$  such that the action of  $b$  is given by

$$b^{-1} a_i b = z_i a_{i+1}, \quad i = 1, \dots, n-1, \quad (2.14)$$

$$b^{-1} a_n b = z_n a_1^{k_1} \cdots a_n^{k_n}. \quad (2.15)$$

Now let  $\zeta_1, \dots, \zeta_n$  be any elements of  $Z$ . By (2.13), the mapping

$$z \mapsto z \quad (z \in Z), \quad a_i \mapsto \zeta_i a_i^{-1}, \quad (2.16)$$

extends to a well-defined automorphism  $\phi$  of  $A$  of order 2. Note that  $\phi$  is trivial on  $Z \supseteq A \cap B$ . Thus, by Lemma 2.11, this extends to an automorphism of  $G$  if  $[\phi, I_b] = 1$ . From (2.14)-(2.16) this reduces to a system of  $n$  equations in the  $n$  unknowns  $\zeta_1, \dots, \zeta_n$ , namely

$$\zeta_i \zeta_{i+1}^{-1} = z_i^2, \quad i = 1, \dots, n-1, \quad (2.17)$$

$$\zeta_1^{-k_1} \cdots \zeta_{n-1}^{-k_{n-1}} \zeta_n^{1-k_n} = z_n^2. \quad (2.18)$$

From (2.10) one deduces that this system has a unique solution. Thus  $G$  possesses an automorphism of order 2, as desired.

CASE 2 : Suppose that  $A^B = \psi_1 \times \psi_2$  where  $\psi_2$  is one-dimensional ( $n_2 = 1$ ) and trivial.

Let  $\psi_1 = \langle I_b \rangle$ . From the previous case we can deduce that there is a basis  $Za_1, \dots, Za_{n-1}, Za = Za_n$  for  $A/(Z \cap A)$  such that

$$a_1^p = a_2^p = \cdots = a_{n-1}^p = 1 \quad (2.19)$$

and that the action of  $I_b$  is given by

$$b^{-1} a_i b = z_i a_{i+1}, \quad i = 1, \dots, n-2, \quad (2.20)$$

$$b^{-1} a_{n-1} b = z_{n-1} a_1^{k_1} \cdots a_{n-1}^{k_{n-1}}, \quad (2.21)$$

$$b^{-1} a b = a, \quad (2.22)$$

for some  $z_1, \dots, z_{n-1}, z \in Z$  and non-negative integers  $k_1, \dots, k_{n-1}$  satisfying (2.10), with  $n$  replaced by  $n - 1$ . The map  $\phi : A \rightarrow A$  should now satisfy

$$z \mapsto z \ (z \in Z), \quad a \mapsto a, \quad a_i \mapsto \zeta_i a_i^{-1} \quad (i = 1, \dots, n - 1). \quad (2.23)$$

We will have a well-defined automorphism of  $A$  provided

$$[a, a_i] = 1, \quad \text{for } i = 1, \dots, n - 1. \quad (2.24)$$

But (2.24) follows from (2.20)-(2.22) and (2.10). For inserting (2.22) into each of the  $n$  equations in (2.20) and (2.21) yields

$$[a, a_i] = [a, a_{i+1}], \quad i = 1, \dots, n - 2, \quad (2.25)$$

$$[a, a_{n-1}] = \prod_{j=1}^{n-1} [a, a_j]^{k_j}. \quad (2.26)$$

Then (2.25) and (2.26), combined with (2.10), yields (2.24). The map  $\phi$  extends to  $G$  for a unique choice of  $\zeta_1, \dots, \zeta_{n-1}$ , as in Case 1. This deals with Case 2.

We now prove Theorem 2.13 in full generality. Suppose  $A^B$  decomposes as in (2.6), where  $\psi_i = \langle I_{b_i} \rangle$  acts irreducibly on the  $n_i$ -dimensional subspace  $W_i$ , and suppose that  $\psi_1, \dots, \psi_s$  are non-trivial. Then for each  $i = 1, \dots, s$  there is a basis  $Za_{i1}, \dots, Za_{in_i}$  for  $W_i$ , and bases  $Za_1, \dots, Za_{r-s}$  for  $W_{s+1}, \dots, W_r$  respectively such that

$$a_{ij}^p = 1, \quad i = 1, \dots, s, \quad j = 1, \dots, n_i, \quad (2.27)$$

and that the action of  $B$  is fully described by

$$b_i^{-1} a_{ij} b_i = z_{ij} a_{i,j+1}, \quad i = 1, \dots, s, \quad j = 1, \dots, n_i - 1, \quad (2.28)$$

$$b_i^{-1} a_{in_i} b_i = z_{in_i} a_{i1}^{k_{i1}} \cdots a_{in_i}^{k_{in_i}}, \quad i = 1, \dots, s, \quad (2.29)$$

$$b_i^{-1} a_{lj} b_i = a_{lj}, \quad \text{whenever } i \neq l, \quad (2.30)$$

$$b^{-1} a_m b = a_m, \quad \forall b \in B, \quad m = s + 1, \dots, r, \quad (2.31)$$

for some  $z_{ij} \in Z$  and non-negative integers  $k_{ij}$  satisfying

$$\sum_{j=1}^{n_i} k_{ij} \not\equiv 1 \pmod{p}, \quad i = 1, \dots, s. \quad (2.32)$$

Then the mapping

$$z \mapsto z \ (z \in Z), \quad a_{ij} \mapsto \zeta_{ij} a_{ij}^{-1}, \quad a_m \mapsto a_m, \quad (2.33)$$

extends, for a unique choice of the elements  $\zeta_{ij} \in Z$ , to an automorphism of  $A$  of order 2, which acts trivially on  $Z$  and commutes with  $A^B$ . Thus  $G$  has an automorphism of order 2 and the theorem is proved.  $\square$

## 3. CASE-BY-CASE ANALYSIS

Throughout this section,  $G$  denotes a potential counterexample to Theorem 1.1, whose non-existence we shall establish. By Lemma 2.8, we have  $\Omega(|\text{Aut}(G)|) \geq 5$  and, together with Corollary 2.10, one readily checks that this leaves the following 19 possibilities for  $|\text{Aut}(G)|$ :

$$3^5 \cdot p, \quad p \in \{5, 7\}, \quad (3.1)$$

$$3^4 \cdot p, \quad p \in \{5, 7, 11, 13, 17, 19, 23\}, \quad (3.2)$$

$$5^4 \cdot 3, \quad (3.3)$$

$$3^4 \cdot 5^2, \quad (3.4)$$

$$3^3 \cdot p^2, \quad p \in \{5, 7\}, \quad (3.5)$$

$$5^3 \cdot 3^2, \quad (3.6)$$

$$3^3 \cdot 5 \cdot p, \quad p \in \{7, 11, 13\}, \quad (3.7)$$

$$3^3 \cdot 7 \cdot 11, \quad (3.8)$$

$$3^2 \cdot 5^2 \cdot 7. \quad (3.9)$$

The order of  $G/Z$  must divide one of those on the above list. Let  $p \in \pi(G/Z)$  and suppose that  $p^2$  does not divide  $|G/Z|$ . Then any Sylow  $p$ -subgroup of  $G$  is abelian. So if  $G/Z$ , and hence  $G$ , has a normal Sylow  $p$ -subgroup, then Lemma 2.5(ii) and Corollary 2.12 imply that  $G$  possesses an automorphism of order 2. Using Lemmas 2.5(i) and 2.7 this fact, together with Lemma 2.9, is easily checked to already rule out all but the following 21 possibilities for  $|G/Z|$ :

$$3^i \cdot 5, \quad i \in \{4, 5\}, \quad (3.10)$$

$$3^i \cdot 13, \quad i \in \{3, 4\}, \quad (3.11)$$

$$3 \cdot 5^i, \quad i \in \{2, 3, 4\}, \quad (3.12)$$

$$3^i \cdot 5^2, \quad i \in \{2, 3, 4\}, \quad (3.13)$$

$$3^i \cdot 7^2, \quad i \in \{1, 2, 3\}, \quad (3.14)$$

$$3^2 \cdot 5^3, \quad (3.15)$$

$$3^i \cdot 5 \cdot 7, \quad i \in \{2, 3\}, \quad (3.16)$$

$$3^i \cdot 5^2 \cdot 7, \quad i \in \{1, 2\}, \quad (3.17)$$

$$3^i \cdot 5 \cdot 11, \quad i \in \{2, 3\}, \quad (3.18)$$

$$3^3 \cdot 5 \cdot 13. \quad (3.19)$$

For these remaining possibilities we may thus assume that  $G/Z$  contains no normal subgroup of prime order. In what follows, there is no loss of generality in assuming that  $\pi(G) = \pi(G/Z)$ . For let  $\pi$  be the set of primes dividing  $|G|$  but not  $|G/Z|$ . By Lemma 2.5, we have  $G = A \times G_1$ , where  $A$  is an abelian Hall  $\pi$ -subgroup of  $G$ . Then, by Corollary 2.12,  $G$  has an automorphism of order 2 unless  $A \cong C_1$  or  $C_2$  and  $\text{Aut}(G) = \text{Aut}(G_1)$ .

As our next step, we claim that in all but three of the above 21 cases, namely

$$3^4 \cdot 5^2, \quad 3^2 \cdot 5^2 \cdot 7, \quad 3^4 \cdot 13, \quad (3.20)$$

we can identify a prime  $p \in \pi(G/Z)$  such that the Sylow  $p$ -subgroup of  $G/Z$  is normal of order  $p^i$ , for some  $i \in \{2, 3, 4, 5\}$ . More precisely we have the following table of values :

$i$	$p$	$ G/Z $
2	3	$3^2 \cdot 5 \cdot 7$ $3^2 \cdot 5 \cdot 11$
2	5	$3 \cdot 5^2$ $3^2 \cdot 5^2$ $3^3 \cdot 5^2$ $3 \cdot 5^2 \cdot 7$
2	7	$3 \cdot 7^2$ $3^2 \cdot 7^2$ $3^3 \cdot 7^2$
3	3	$3^3 \cdot 13$ $3^3 \cdot 5 \cdot 7$ $3^3 \cdot 5 \cdot 11$ $3^3 \cdot 5 \cdot 13$
3	5	$3 \cdot 5^3$ $3^2 \cdot 5^3$
4	3	$3^4 \cdot 5$
4	5	$3 \cdot 5^4$
5	3	$3^5 \cdot 5$

In 11 of the 18 cases listed in the table, the normality of the identified  $p$ -subgroup is established by a direct application of Lemma 2.5(i). In the case of  $3^3 \cdot 13$  we also need to use our assumption that  $G_{13}$  is not normal, being of prime order. The remaining 6 cases are those where  $\omega(|G|) = 3$ . Here we appeal both to the aforementioned assumption and to Lemma 2.7. One example will suffice to illustrate the idea, say  $3^2 \cdot 5 \cdot 7$ . By assumption neither  $G_5$  nor  $G_7$  is normal. Thus, by Lemma 2.7, some 3-subgroup must be characteristic. If this is  $G_3$ , we are done, otherwise it is isomorphic to  $C_3$ . The quotient group then has order  $3 \cdot 5 \cdot 7$ . Applying Lemma 2.7 to the quotient, we find that  $G/Z$  has a characteristic subgroup of order 9, 15 or 21. In the first instance we are done. In the latter two, Lemma 2.5(i) and eq. (2.4) imply that  $G_5$  resp.  $G_7$  is characteristic after all, contradicting our assumptions.

Of the three numbers (3.20) omitted from the table, we can still apply Lemma 2.7 as above to conclude in the case of both  $3^4 \cdot 5^2$  and  $3^2 \cdot 5^2 \cdot 7$  that either the  $G_3$  or  $G_5$ -subgroup must be normal, though we don't know which one a priori. This leaves  $3^4 \cdot 13$ , where applying the same kind of analysis allows us only to conclude that either  $G_3$  or a subgroup of order  $3^3$  must be normal. The case when  $|G/Z| = 3^4 \cdot 13$  will be called the *exceptional* case.

In the next part of our analysis, we ignore the exceptional case. This will be dealt with at the finish. In each of the remaining 20 cases, we are guaranteed to have a splitting  $G = P \rtimes B$ , where

- (i)  $P$  is a Sylow  $p$ -subgroup for some prime  $p \in \{3, 5, 7\}$ ,
- (ii)  $|P/(Z \cap P)| = p^i$  for some  $i \in \{2, 3, 4, 5\}$ ,
- (iii)  $B$  is a  $p'$ -group, either abelian or of class 2,
- (iv) If  $p = 3$  then  $\pi(B) \subseteq \{5, 7, 11, 13\}$ ,
- (v) If  $p = 5$  then  $i \leq 4$  and  $\pi(B) \subseteq \{3, 7\}$ ,
- (vi) If  $p = 7$  then  $i = 2$  and  $\pi(B) = \{3\}$ .

We now divide the analysis into 2 steps, according as to whether  $P/(Z \cap P)$  is elementary abelian or not.



STEP 1 :  $P/(Z \cap P)$  is elementary abelian.

If  $B$  acts trivially on  $P/(Z \cap P)$  then  $G = P \times B$ . It is easy to check, using the tools of Section 2, that in all cases  $B$  possesses an automorphism of order 2, hence so does  $G$ . If  $B$  acts non-trivially then all the conditions of Theorem 2.13 are satisfied, so we are done here too.

STEP 2 :  $P/(Z \cap P)$  is not elementary abelian.

Then  $P/(Z \cap P)$  must have order  $p^3$  at least. In all but 4 cases, namely

$$5^3 \cdot 3, \quad 5^3 \cdot 3^2, \quad 5^4 \cdot 3, \quad 3^5 \cdot 5 \quad (3.21)$$

we can apply Lemmas 2.4 and 2.6 directly to find that  $B$  acts trivially on  $P$ , and then argue as in Step 1. We illustrate with the example of  $3^4 \cdot 5$ . By Lemma 2.3(ii), there is a proper subgroup  $P_2$  of  $P$ , strictly containing  $Z \cap P$ , and such that  $P_2/(Z \cap P) \subseteq Z[P/(Z \cap P)]$ . Thus  $P_2$  is invariant under the action of  $B$ , so  $B$  acts on both  $P_2/(Z \cap P)$  and  $P/P_2$ , and both these groups are of order  $3^i$  for some  $i < 4$ . Now Lemma 2.6 implies that  $|\mathrm{GL}(i, 3)|$  is not divisible by 5 for any  $i < 4$ . It follows that  $B$  acts trivially on both  $P_2/(Z \cap P)$  and  $P/P_2$ , and hence on  $P/(Z \cap P)$  by Lemma 2.4.

This leaves us with the four cases in (3.21). First suppose  $|G/Z| \in \{5^3 \cdot 3, 5^3 \cdot 3^2\}$ . In both cases, we can choose a characteristic subgroup  $P_2$  of  $P$  as above such that  $|P_2/(Z \cap P)| = 5$ . Then  $B$  centralises  $P_2$ , since  $|\mathrm{GL}(1, 5)|$  is not divisible by 3. But  $C_G(P_2)$  must then be a proper, characteristic subgroup of  $G$  containing  $P_2$  and  $B$ , and hence of index dividing  $5^2$ . Thus  $G' \subseteq C_G(P_2)$ , which means that  $B$  must also act trivially on  $P/P_2$ , thus on all of  $P/(Z \cap P)$ , by Lemma 2.4.

If  $|G/Z| = 5^4 \cdot 3$  and  $P/(Z \cap P)$  is non-abelian we can reach the same conclusion, for either the commutator subgroup or center of  $P/(Z \cap P)$  must have order 5 and yields the characteristic subgroup playing the role of  $P_2$  above. If  $P/(Z \cap P)$  is abelian, it must be isomorphic to  $C_{25} \times C_{25}$ . In this case, we can still use the method of Theorem 2.13. For one easily checks that  $3 \parallel |\mathrm{Aut}(C_{25} \times C_{25})|$ , and that if  $B/(Z \cap B) := \langle Zb \rangle$  acts non-trivially on  $P/(Z \cap P)$ , then there is a generating set  $Za_1, Za_2$  for the latter such that

$$b^{-1}a_1b = a_2, \quad b^{-1}a_2b = za_1^{-1}a_2^{-1}, \quad \exists z \in Z. \quad (3.22)$$

This was the starting point of the method of proving Case 1 of Theorem 2.13, and the same argument carries through here.

Finally, suppose  $|P/(Z \cap P)| = 3^5 \cdot 5$ . Let  $P_2/(Z \cap P)$  be a non-trivial characteristic subgroup of  $P/(Z \cap P)$  of index at least  $3^2$ , whose existence is guaranteed by Lemma 2.3(ii). As noted previously,  $|\mathrm{GL}(i, 3)|$  is not divisible by 5, for any  $i < 4$ . Hence we can deduce from Lemma 2.3 that  $B$  acts trivially on  $P$ , unless  $|P_2/(Z \cap P)| = 5$  and  $B$  acts irreducibly on  $P/P_2$ . But the latter is impossible, since  $P_2B$  is contained in the characteristic subgroup  $C_G(P_2)$  of  $G$ . This completes Step 2.

All that now remains in order to establish Theorem 1.1 is to handle the exceptional case where  $|G/Z| = 3^4 \cdot 13$ . We can assume that the Sylow 3-subgroup of  $G$  is not

normal, as otherwise Theorem 2.13 can be applied directly. As noted earlier, some 3-subgroup  $P$  such that  $|P/(Z \cap P)| = 3^3$  must then be normal. Since 13 does not divide  $|\mathrm{GL}(2, 3)|$ , then  $P/(Z \cap P)$  must be elementary abelian and acted upon irreducibly by any Sylow 13-subgroup, or else the latter would also be normal in  $G$ . Moreover, the quotient  $G/P$  must be non-abelian of order 39, isomorphic to  $N_G(Q)/(Z \cap Q)$ , for any Sylow 13-subgroup  $Q$  of  $G$ . These facts combined imply that

- (i)  $G/Z$  is a centerless group,
- (ii)  $|G/G'|$  is a multiple of 3,
- (iii)  $G = PB$  for some subgroup  $B$  with  $P \cap B \subseteq Z$ .

By Lemma 2.1, (i) implies that any non-trivial central automorphism of  $G$  is outer. But if  $G$  possesses any outer automorphisms, then  $|\mathrm{Aut}(G)|$  will have to be at least  $3^5 \cdot 13$ , which is greater than  $3^7$ , a contradiction. Thus  $G$  cannot have any non-trivial central automorphisms. But then (ii) and Lemma 2.2 force the conclusion that  $Z$  contains no elements of order 3. This and (iii) in turn imply that  $G = P \rtimes B$  and that  $P$  is elementary abelian of order 27. But now we can apply Corollary 2.12 to conclude that  $G$  has an automorphism of order 2. The theorem is proved.

#### 4. DISCUSSION

A careful reading of the foregoing proof will show that we have established somewhat more than we have stated. In fact, if  $G/Z$  has odd order less than  $3^7$ , then either  $G$  has an automorphism of order 2, or  $G/Z$  is a group of order  $3^4 \cdot 13$  with a very specific structure : namely, it is the split extension of an elementary abelian group of order  $3^3$  by a non-abelian group of order 39. In the latter case we could only show that either  $G$  has an automorphism of order 2, or a non-trivial central automorphism which was necessarily outer. It would be interesting to establish if this case is a genuine exception or whether a group with this structure cannot be N.I. either. This might in turn shed light on the following questions :

1. What is the smallest order of a non-nilpotent N.I. group and, conversely, the smallest order of a non-nilpotent automorphism group of odd order ? Theorem 1.1 and the result in [MS] give the answers for nilpotent groups. Martin's result [M] that almost all  $p$ -groups have automorphism group a  $p$ -group also motivates giving special attention to non-nilpotent N.I. groups.
2. It should be particularly interesting to study complete groups, as these have no non-trivial outer or central automorphisms. Several papers in the literature deal with the construction of complete groups of odd order (see [D], [H], [HR], [Sch] and [So] for example), but the smallest possible order of such a group is unknown. The smallest example in the literature seems to be a complete group of order  $3^{12} \cdot 5$  constructed in [H].
3. In [SZ] the authors conjectured that if  $\Omega(|\mathrm{Aut}(G)|) \leq 5$  then  $|\mathrm{Aut}(G)|$  is even (recall Lemma 2.8 above). This problem remains open. Resolving it would also shed light on the previous two questions.

Finally, we remark that Corollary 2.12 and Theorem 2.13 are of some interest in their

own right. They give sufficient conditions for an automorphism of order 2 of a normal subgroup to be extendible to the whole group, when the subgroup is nilpotent of class at most 2. It would be interesting to find similar sufficient conditions when the subgroup has class 3.

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