Homework 1 (due Monday, Nov. 28)

Q.1. At a recent gathering of Göteborgs Internationella Socialister, there were n people present, and they were a mixture of Stalinists and Trotskyists. Each Stalinist had at least one personal enemy amongst the Trotskyists. Show that there must be a subset $\mathcal{S}$ consisting of at least $n/2$ of those present, such that each Stalinist in $\mathcal{S}$ had an odd number of Trotskyist enemies in $\mathcal{S}$.

(NOTE : The result holds for any n, though of course in this example n is small).

Q.2 Let $\mathcal{F}$ be a family of subsets of $\{1, 2, ..., n\}$ in which no set is a proper subset of any other. Show that the family $\mathcal{F}$ contains no more than $\binom{n}{\lfloor n/2 \rfloor}$ sets.

(HINT : Consider a random permutation π of $\{1, 2, ..., n\}$ and the random variable $X = \#\{i : \{\pi(1), ..., \pi(i)\} \in \mathcal{F}\}$.)

Q.3 (i) Prove that, if there exists a real number $p \in [0, 1]$ such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{l} (1-p)^{\binom{l}{2}} < 1,$$

then the Ramsey number $R(k, l)$ satisfies $R(k, l) > n$.

(ii) Deduce that

$$R(4, l) = \Omega\left(\left(\frac{l}{\log l}\right)^{3/2}\right).$$

Q.4 Let k, n be positive integers with $k \leq n$. An *increasing sequence of length* k in a permutation π of $\{1, ..., n\}$ is a sequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$.

With this definition in mind, prove the following : There exists an absolute constant $c > 0$ with the following property. Let A be an $n \times n$ matrix with distinct real entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length $c\sqrt{n}$.

(NOTE : By "absolute" we mean that the constant does not depend on n. For full marks, give an explicit constant.)

Q.5 Let $\epsilon \in (0,1)$, $n \in \mathbb{N}$ and let G be a graph on n vertices such that every vertex has degree at least n^ϵ. Prove that there exists an absolute constant $c > 0$ for which it is possible to partition the vertices into two sets A and B such that the following two properties hold :

(i) $|A| \leq c_\epsilon n^{1-\epsilon} \log n$,

(ii) Every vertex in B has at least one neighbour in each of A and B.

(HINT : Create A randomly by putting each vertex in A with probability p (where p has to be chosen intelligently). Let B of course be the complement of A. Show that, for an appropriate choice of p, the probability that the partition satisfies the conditions required is positive and hence there is such a partition.)

Q.6 Let $\mathcal{F}$ be a finite collection of binary strings of finite lengths and assume no string is a prefix of any other. For each $i \in \mathbb{N}$, let N_i denote the number of strings of length i in $\mathcal{F}$. Prove that

$$\sum_{i=1} \frac{N_i}{2^i} \leq 1.$$

(HINT : If $A_1, A_2, ...$ are mutually exclusive events in a probability space, then $\sum_i \mathbb{P}(A_i) \leq 1$).

Q.7 Let X be a real-valued random variable. Prove that

$$\mathbb{P}(X = 0) \leq \frac{\text{Var}(X)}{\mathbb{E}[X^2]},$$

assuming both quantities on the right-hand side are finite.

Q.8 (i) Let X be a random variable with mean zero and finite variance. Show that, for all $\lambda > 0$,

$$\mathbb{P}(X \geq \lambda) \leq \frac{\text{Var}(X)}{\text{Var}(X) + \lambda^2}.$$

SUGGESTION : Solve the problem in the following steps :

(a) Show that it suffices to show the result when the variance is 1.

(b) Assuming now that the variance is 1, for each positive c, obtain a lower bound on $\mathbb{E}[(X + c)^2]$ involving $\mathbb{P}(X \geq \lambda)$, by looking only when $X \geq \lambda$.

(c) Optimize the choice of c in the inequality obtained in step 2 to prove the result.

(ii) (This part can be done without having done part **(i)**) If X has a symmetric distribution (meaning that $\mathbb{P}(X > \lambda) = \mathbb{P}(X < -\lambda)$ for each λ), discuss what, if anything, the result of part **(i)** gives you beyond what Chebyshev's inequality would.

Q.9 Let $\mathbf{v}_1 = (x_1, y_1), \mathbf{v}_2 = (x_2, y_2), ..., \mathbf{v}_n = (x_n, y_n)$ be n two-dimensional integer vectors, where each x_i and y_i has an absolute value not exceeding $\frac{2^{n/2}}{100\sqrt{n}}$. Prove that there must exist a pair I, J of disjoint subsets of $\{1, ..., n\}$, not both empty, such that

$$\sum_{i \in I} \mathbf{v}_i = \sum_{j \in J} \mathbf{v}_j.$$