On $m$-covering families of Beatty sequences with irrational moduli

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If $A$ is a set and $m \in \mathbb{N}$, we denote by $mA$ the multiset union of $m$ copies of $A$. 
Beatty’s Theorem
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Let \( \alpha, \beta \) be positive real numbers. Then

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\{ \lfloor n\alpha \rfloor : n \in \mathbb{N} \} \cup^* \{ \lfloor n\beta \rfloor : n \in \mathbb{N} \} = \mathbb{N}
\]

if and only if \( \alpha \) and \( \beta \) are irrational and satisfy

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\frac{1}{\alpha} + \frac{1}{\beta} = 1.
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Wythoff Nim

The $P$-positions of Wythoff Nim are the pairs

$$\{\{[n\phi], [n\phi^2]\} : n \in \mathbb{N}_0\},$$

where

$$\frac{1}{\phi} + \frac{1}{\phi^2} = 1 = \phi^2 - \phi \Rightarrow \phi = \frac{1 + \sqrt{5}}{2}. $$
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$$\pi([n\phi]) = [n\phi^2], \quad \pi = \pi^{-1},$$

has the property that

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I won’t pursue this line any further here, but see my paper with U. Larsson:


$k$-Wythoff Nim
\textbf{k-Wythoff Nim}

One allows diagonal moves \((x, y) \rightarrow (r, s)\), where \(x \geq r, y \geq s, \max\{x - r, y - s\} > 0\) and

\[| (x - r) - (y - s) | < k. \]
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\[|(x - r) - (y - s)| < k.\]

Then Fraenkel showed that the \(P\)-positions are given by the pairs \(\{[[nr_k], [ns_k]] : n \in \mathbb{N}_0\}\), where

\[\frac{1}{r_k} + \frac{1}{s_k} = 1, \quad s_k - r_k = k,\]
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\[ \Rightarrow r_k = \frac{2 - k + \sqrt{k^2 + 4}}{2}. \]
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Let $m \in \mathbb{N}$. Let $\alpha, \beta$ be positive real numbers. Then

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It was rediscovered by Larsson (2009) in his study of various generalisations of $k$-Wythoff Nim, including a certain blocking game.
We shall henceforth employ the following notation for Beatty sequences:

\[ S(\alpha, \beta) := \{\lfloor n\alpha + \beta \rfloor : n \in \mathbb{N}\}. \]
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**Upsensky’s Theorem (1927)**

Let \( \alpha_1, \ldots, \alpha_k \) be positive real numbers. Then

\[ S(\alpha_1, 0) \cup^* \cdots \cup^* S(\alpha_k, 0) = \mathbb{N} \]

if and only if either

(i) \( k = 1 \) and \( \alpha_1 = 1 \), or

(ii) \( k = 2 \), \( \alpha_1 \not\in \mathbb{Q} \) and \( 1/\alpha_1 + 1/\alpha_2 = 1 \).
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This result indicates that it might not be so easy to generalise Wythoff Nim to more than 2 piles in a ‘nice’ way. Fraenkel has proposed a generalisation to any number of piles, but these games are still not well understood.
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**Theorem (H, 2010)**

Let $m \in \mathbb{N}$. Let $\alpha_1, \ldots, \alpha_k$ be positive irrational numbers satisfying

$$\sum_{i=1}^{k} \frac{1}{\alpha_i} = m.$$ 

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Then

$$S(\alpha_1, 0) \cup^* \cdots \cup^* S(\alpha_k, 0) = m\mathbb{N}$$

if and only if:

- $k$ is even, $k = 2l$ say, and the $\alpha_i$ can be reordered so that

  $$\frac{1}{\alpha_{2i-1}} + \frac{1}{\alpha_{2i}} \in \mathbb{Z}, \quad i = 1, \ldots, l.$$
Idea of Proof

- Set $\theta_i := 1/\alpha_i$. The sequences form an eventual exact $m$-cover if and only if the density condition is satisfied and the function $\epsilon: \mathbb{N} \to \mathbb{N}$ defined by

$$
\epsilon(n) := \sum_{i=1}^{k} \{n\theta_i\}, \quad \text{(here } \{x\} := x - \lfloor x \rfloor)$$

is ultimately constant.
- Weyl equidistribution is then used to reduce the problem to the following arithmetical fact:
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**Proposition**

Let $a_1, ..., a_\mu, c_1, ..., c_\nu$ be positive integers and $b_1, ..., b_\mu, d_1, ..., d_\nu$ be any integers. If, for every $t \in \mathbb{Z}$, we have an equality of multisets

$$\bigstar \bigcup_{i=1}^{\mu} S(a_i, tb_i) = \bigstar \bigcup_{j=1}^{\nu} S(c_j, td_j),$$

then $\mu = \nu$, and we can reorder so that, for each $i = 1, ..., \mu$, $a_i = c_i$ and $b_i \equiv d_i \pmod{a_i}$. 
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> For distinct rational moduli, there is the famous **Tiling Conjecture** of Fraenkel, which states that if \( k \geq 3 \) and the \( \alpha_i \) are distinct positive rationals, then

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\star \bigcup_{i=1}^{k} S(\alpha_i, \beta_i) = \mathbb{N} \iff \alpha_i = \frac{2^k - 1}{2^{k-i}}, \quad i = 1, \ldots, k.
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Fraenkel has constructed various multi-pile subtraction games (e.g.: Rat Game) whose \( P \)-positions correspond to complementary rational Beatty sequences.
For $m = 1$, Graham (1973) reduced the classification of eventual exact $m$-covers of $\mathbb{N}$ by Beatty sequences at least one of whose moduli is irrational, to the integer moduli case.
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Our approach extends the classification in a sense to $m > 1$, but the arithmetical structure is more complicated than in the homogeneous case.
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The first step would be to have a ‘fractional version’ of Beatty’s theorem, i.e.: a version for pairs $\{\alpha_1, \alpha_2\}$ of positive irrationals satisfying

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = m,$$

where now $m \in \mathbb{Q}$. 
**Fraenkel**: Can one construct (invariant) subtraction games whose $P$-positions correspond, in some sense, to ‘fractional Beatty sequences’?

The first step would be to have a ‘fractional version’ of Beatty’s theorem, i.e.: a version for pairs $\{\alpha_1, \alpha_2\}$ of positive irrationals satisfying

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where now $m \in \mathbb{Q}$.

Such a result is given in my paper. It is technical to state in full, but the main points are the following:
Let $\alpha_1, \alpha_2$ be positive irrationals satisfying

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$$r(n) := \#\{ k : [k\alpha_1] = n \} + \#\{ k : [k\alpha_2] = n \}.$$
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- If $q = 2$, then $r(n) \in \{[p/2], \lceil p/2 \rceil\}$ for every $n \in \mathbb{N}$. 

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- If $q = 2$, then $r(n) \in \{[p/2], \lceil p/2 \rceil\}$ for every $n \in \mathbb{N}$.
- If $q > 2$, then either

  $$r(n) \in \{[p/q], \lceil p/q \rceil, \lceil p/q \rceil + 1\}, \quad \text{for every } n \in \mathbb{N}$$

  or

  $$r(n) \in \{[p/q] - 1, [p/q], \lceil p/q \rceil\}, \quad \text{for every } n \in \mathbb{N},$$

  depending on the values of $\{1/\alpha_1\}$ and $p \pmod{q}$. 

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depending on the values of $\{1/\alpha_1\}$ and $p \pmod{q}$.
Moreover, the densities of the sets on which $r(n)$ is constant
can be computed.