Att lämna in onsdag den 1 oktober

1. Below we shall define three sequences (R_n), (S_n), (T_n) of positive integers. Your task is to show that, for all $n > 0$,

$$R_n = S_n = T_n = C_n,$$

where C_n is the n:th Catalan number.

(a) Let R_n denote the number of sequences $a_1 a_2 \cdots a_n$ of n positive integers which are non-decreasing, i.e.:

$$a_i \leq a_{i+1}, \quad i = 1, ..., n-1,$$

and such that

$$a_i \leq i, \quad i = 1, ..., n.$$

For example, $R_3 = 5$ since there are the following five allowed sequences of length three :

111 112 113 122 123

(b) S_n är antalet sätt att dela upp en regulär $(n+2)$-hörning i n st. triangler genom att rita in $n-1$ st. diagonala linjer.

(OBS! If you don't understand what I mean, ask me or one of the other övningsledarna for further explanation).

(c) T_n is the number of n-tuples $(x_1, ..., x_n)$ of non-negative integers such that

$$0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq n,$$

and

$$n+1 \text{ divides } \sum_{i=1}^{n} x_i.$$

HINT : For parts **(a)** and **(b)** it is probably easiest to show that R_n and S_n both satisfy the same recurrence relation as that for C_n which we learned in class. However, for part **(c)** it is probably easier to prove directly that

$T_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$.

2. Find a formula similar to that in demonstration exercise no. 2, Wednesday, Sept. 17, for the number A_n of ways that n married couples can stand in a line so that noone is standing next to their spouse.

3. Prove that the only integer solutions to the equation

$$x^4 - 2y^2 = 1$$

are $x = \pm 1,\ y = 0$.
(Hint : Fundamental theorem of atirhmetic).

4. On the other hand, prove that the equation

$$x^2 - 2y^2 = 1$$

has infinitely many integer solutions.
(Hint : As you know from studying complex numbers, $z\overline{z} = |z|^2$ when $z = x + \sqrt{-1}y$ for real numbers x, y).

5. Define the function $\mu : \mathbf{N} \to \mathbf{N}$ as follows :

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ DISTINCT primes,} \\ 0, & \text{otherwise.} \end{cases}$$

A famous theorem of Euler says that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad (1)$$

Using (1), compute

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}.$$