Homework 2 : Solutions

1 (a) The number of sequences of length n we're denoting R_n. Clearly, $R_0 = C_0 = 1$, since the empty sequence is allowed. It remains to show that, for all $n > 0$,

$$R_n = \sum_{m=1}^{n} R_{m-1} R_{n-m}.$$

FIRST SOLUTION :

For $m = 1, ..., n-1$ let $R_n(m)$ denote the number of sequences of length n such that

$$a_{m+1} = m+1, \quad a_i < i \text{ for } i = 2, ..., m. \tag{1}$$

Let $R_n(n)$ denote the number of sequences for which

$$a_i < i \text{ for all } i = 2, ..., n. \tag{2}$$

Then clearly

$$R_n = \sum_{m=1}^{n} R_n(m), \tag{3}$$

so we'll be done if we can show that

$$R_n(m) = R_{m-1} R_{n-m}, \quad m = 1, ..., n. \tag{4}$$

First suppose $1 \leq m \leq n-1$ and let $a_1 \cdots a_n$ be one of the $R_n(m)$ sequences satisfying (1). Then $a_2 = 1$ and the sequence $a_2 a_3 \cdots a_m$, of length $m-1$, must satisfy exactly the same requirements as at the outset. Hence, there are R_{m-1} possibilities for it. Next, for $i = m+1, m+2, ..., n$ let $b_i = a_i - m$. Then the sequence $b_{m+1} b_{m+2} \cdots b_n$, of length $n-m$, satisfies exactly the saem requirements as at the outset, so there are R_{n-m} possibilities for it, and hence also for $a_{m+1} a_{m+2} \cdots a_n$.

Thus, by MP, there are $R_{m-1} R_{n-m}$ possibilities for the whole sequence $a_1 \cdots a_n$, which proves (4) in the case $1 \leq m \leq n-1$.

Finally, then, we consider the $R_n(n)$ sequences satisfying (2). These are the sequences for which $a_2 = 1$ and the sequence $a_2 \cdots a_n$, of length $n-1$, satisfies exactly the same conditions as at the outset. Hence there are $R_{n-1} = R_{n-1}R_0$ possibilities, and so (4) is verified even in this case.

SECOND SOLUTION :

Instead, for $m = 1, ...n$, define $R_n(m)$ to be the number of sequences of length n such that

$$a_m = m, \quad a_i < i \text{ for all } i > m. \tag{5}$$

Clearly, (3) holds and so it again suffices to prove (4) for each m.

So let $a_1 \cdots a_n$ be one of the $R_n(m)$ sequences satisfying (5). The left-subsequence $a_1 \cdots a_{m-1}$, of length $m-1$, must satisfy exactly the same conditions as at the outset and hence there are R_{m-1} possibilities for it.

To deal with the right-subsequence $a_{m+1} \cdots a_n$, let $b_i = a_i - (m-1)$ for $i = m+1, ..., n$. Then the condition (5) implies that $b_1 = 1$ and that the whole sequence $b_{m+1} \cdots b_n$, of length $n-m$, satisfies exactly the same conditions as at the outset. Hence there are R_{n-m} possibilities for it, and hence also for $a_{m+1} \cdots a_n$.

Finally, an application of MP verifies (4).

(b) There is exactly one way to divide a line segment (a 2-gon) into zero triangles, namely do nothing, hence $S_0 = C_0 = 1$. Hence it suffices to prove that, for all $n > 0$,

$$S_n = \sum_{m=1}^{n} S_{m-1}S_{n-m}. \tag{6}$$

Think of our $(n+2)$-gon as inscribed in a circle. Fix two adjacent vertices, call them A and B, where B is clockwise from A. Moving clockwise from B, label the remaining n vertices with the integers $1, ..., n$.

Now, for $m = 1, ..., n$, let $S_n(m)$ denote the number of triangulations of our $(n+2)$-gon which include the triangle $\{A, B, m\}$. Since the edge $\{A, B\}$ must be included in SOME triangle, it is clear that

$$S_n = \sum_{m=1}^{n} S_n(m).$$

Hence, it suffices to prove that

$$S_n(m) = S_{m-1}S_{n-m}, \quad m = 1, ..., n. \tag{7}$$

But this is easy. The triangle $\{A, B, m\}$ divides the remainder of the $(n+2)$-gon into 2 smaller regions, call them X and Y, where X is to the left of the triangle and Y to the right. X is an $(n-m+2)$-gon consisting of the vertices $m, m+1, ..., n, A$, hence there are S_{n-m} ways to triangulate it. Similarly, Y is an $(m+1)$-gon consisting of the vertices $B, 1, 2, ...m$, hence can be triangulated in S_{m-1} ways. By MP, there are thus $S_{m-1}S_{n-m}$ ways to triangulate both X and Y, in other words, to triangulate the entire $(n+2)$-gon so that the triangle $\{A, B, m\}$ appears. This proves (7).

(c) First we prove that there are $\binom{2n}{n}$ possible n-tuples $(x_1, ..., x_n)$ satisfying only the requirement that

$$0 \le x_1 \le \cdots \le x_n \le n. \tag{8}$$

Indeed, there is a simple 1-1 correspondence between these n-tuples and the n-element subsets of $\{1, ..., 2n\}$ given by

$$(x_1, ..., x_n) \leftrightarrow \{y_1, ..., y_n\},$$

where

$$x_1 := y_1 - 1,$$
$$x_k := x_{k-1} + (y_k - y_{k-1} - 1), \quad k = 2, ..., n.$$

When an integer is divided by $n+1$, there are $n+1$ possibilities for the remainder, namely $0, 1, ..., n$. Hence we'll be done if we can show that, amongst all the n-tuples satisfying (8), the remainders modulo $n+1$ left by the sums $\sum_{i=1}^{n} x_i$ are equidistributed.

This is also easy, for we may describe, for each $r = 1, ..., n$, an explicit 1-1 correspondence between the n-tuples $(x_1, ..., x_n)$ such that $\sum x_i \equiv r \pmod{n+1}$ and those for which $\sum x_i \equiv r - 1 \pmod{n+1}$.

The correspondence is described as follows : an n-tuple $(x_1, ..., x_n)$ for which $\sum x_i \equiv r$ is first taken to

$$(x_1 \oplus 1, x_2 \oplus 1, ..., x_n \oplus 1),$$

where $\oplus$ denotes addition modulo $n+1$. Then the coordinates are rearranged, if necessary, so that (8) holds. One readily checks that

$$\sum x_i \oplus 1 \equiv \sum x_i \oplus \sum 1 \equiv r + n \equiv r - 1 \pmod{n+1}, \quad \text{v.s.v.}$$

2. Let X be the set of all permutations of the $2n$ people. For $i = 1, ..., n$ let A_i denote the set of all those permutations in which the i:th married couple stand next to one another. Then we want to compute

$$\left| X \backslash \left(\bigcup_{i=1}^{n} A_i \right) \right| .$$

We do this using the inclusion-exclusion principle, which says that

$$\left| X \backslash \left(\bigcup_{i=1}^{n} A_i \right) \right| = |X| - \sum_{i=1}^{n} |A_i| \tag{9}$$
$$+ \sum_{i \neq j} |A_i \cap A_j| - \sum_{i \neq j \neq k} |A_i \cap A_j \cap A_k|$$
$$+ \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

A typical term on the rhs of (9) is

$$(-1)^k \cdot |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \tag{10}$$

where $i_1 \neq i_2 \neq \cdots \neq i_k$ and $0 \leq k \leq n$. Apart from the $(-1)^k$ factor, this counts the number of permutations for which a specificed k of the n couples are put together. We claim that

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = 2^k (2n - k)! \tag{11}$$

We explain (11) as follows : first we may permute as we like $2n - k$ 'objects', one for each of the k 'glued' couples, and one for each of the remaining $2n - 2k$ individuals. There are $(2n - k)!$ ways to do this. We still have to decide, for each of the k glued couples, who'll stand to the left and whom to the right. There are thus 2 choices left for each such couple, hence (by MP) 2^k choices in all for this final step. Another application of MP verifies (11). From (11) we are lead to directly to the result that

$$\left| X \backslash \left(\bigcup_{i=1}^{n} A_i \right) \right| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k (2n - k)!$$

For one simply has to note that, for each $k = 0, ..., n$, the factor $\binom{n}{k}$ arises since there are so many terms of the form (10) on the rhs of (9), one for each choice of k couples from n.

3. Let (x, y) be an integer solution to

$$x^4 - 1 = 2y^2. \tag{12}$$

We will show that $x = \pm 1$, which immediately implies that $y = 0$. First, since the HL of (12) is even, so is the VL, hence x^4 is odd, hence so is x. Let us now write (12) as

$$(x^2 - 1)\left(\frac{x^2+1}{2}\right) = y^2. \tag{13}$$

Note that, since x is odd, both factors on the VL of (13) are integers. We claim that these two factors are relatively prime. So let d be a common divisor of $x^2 - 1$ and $(x^2+1)/2$. Then d also divides x^2+1 and hence divides $(x^2+1)-(x^2-1) = 2$. Hence d is either 1 or 2. To prove our claim it therefore suffices to show that $(x^2+1)/2$ is an odd number. This is equivalent to x^2+1 not being divisible by 4. But x, being odd, is $\equiv \pm 1 \pmod 4$. Hence $x^2 + 1 \equiv (\pm 1)^2 + 1 \equiv 1 + 1 \equiv 2 \not\equiv 0 \pmod 4$, v.s.v.

So we've established the claim that the two factors on the VL of (13) are relatively prime. But their product, being equal to y^2, is a perfect integer square. Hence, FTA implies that each factor is itself a perfect square. In other words, there exist integers z, w such that

$$x^2 - 1 = z^2, \qquad \frac{x^2+1}{2} = w^2.$$

But it is clear that the only solutions to the first equation above are $x = \pm 1$, $z = 0$, since this is the only way two integer squares can differ by 1. So we're done !

4. One integer solution to the equation

$$x^2 - 2y^2 = 1 \tag{14}$$

is $x_0 = 3$, $y_0 = 2$. Let (x_n, y_n) be any integer solution. Then so is (x_{n+1}, y_{n+1}) where

$$x_{n+1} = x_n^2 + 2y_n^2, \qquad y_{n+1} = 2x_n y_n. \tag{15}$$

For a little algebra shows that, for any variables A, B we have

$$(A^2 - 2B^2)^2 = (A^2 + 2B^2)^2 - 2 \cdot (2AB)^2. \tag{16}$$

Thus, if we take $A = x_n$, $B = y_n$, so that $A^2 - 2B^2 = 1$, then the VL of (16) is also equal to 1.

Starting from $(x_0, y_0) = (3, 2)$ and iterating the recurrence (15), we get infinitely many distinct solutions, since it is clear that if $x_n > 1$, $y_n > 1$, then $x_{n+1} > x_n$ and $y_{n+1} > y_n$.

As good pedegogy, we note that (15) doesn't need to be pulled out of a hat - there's an idea behind it. Namely, $A^2 - 2B^2$ can be factorised as

$$A^2 - 2B^2 = (A + \sqrt{2}B)(A - \sqrt{2}B). \tag{17}$$

We think of the HL of (17) as being of the form $z\bar{z}$, where

$$\overline{A + \sqrt{2}B} \stackrel{\text{def}}{=} A - \sqrt{2}B,$$

whenever A, B are integers (rational numbers are ok, too, but not anything involving $\sqrt{2}$). Numbers of the form of z can be multiplied together, and one gets back numbers of the same form. One may also check by direct computation that, for any two such numbers z_1, z_2, one has

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}.$$

Thus, squaring (17), we get

$$(A^2 - 2B^2)^2 = (z\overline{z})^2 = z^2(\overline{z})^2 = z^2\overline{z^2}. \tag{18}$$

A direct computation gives

$$z^2 = (A + \sqrt{2}B)^2 = (A^2 + 2B^2) + \sqrt{2}(2AB). \tag{19}$$

From (19), (18) and (17), we deduce (15).

5. One observes that

$$\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{a_n}{n^2}, \tag{20}$$

where

$$a_n = \sum_{d|n} \mu(d).$$

I claim that

$$\sum_{d|n} \mu(d) = 0, \quad \text{for all } n > 1. \tag{21}$$

Since $\mu(1) = 1$, (21) implies that the HL of (20) equals simply 1, and hence that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}.$$

So it remains to prove (21). Let n be an integer greater than 1 and let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}, \quad (m \geq 1),$$

be its' prime factorisation. The only divisors d of n for which $\mu(d) \neq 0$ are those which are products of distinct primes. For each $k = 0, ..., m$ there are $\begin{pmatrix} m \\ k \end{pmatrix}$ such divisors which are products of exactly k distinct primes. Each of these contributes $(-1)^k$ to the sum in (21). Hence

$$\sum_{d|n} \mu(d) = \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix}. \tag{22}$$

But we recall that it is a consequence of the binomial theorem that the HL of (22) is equal to zero for any integer $m \geq 1$. So we're done !