Homework 3 : Solutions

1 (i) Here's the coloring of continental Europe provided by the greedy algorithm, with a certain choice of countries, and english alphabetical order :

Albania	1	Germnay	3	Portugal	1
Austria	1	Greece	2	Romania	2
Belgium	1	Hungary	3	Serbia	4
Bosnia	1	Italy	2	Slovakia	4
Bulgaria	1	Liechtenstein	2	Slovenia	4
Croatia	2	Luxembourg	4	Spain	3
Czech Rep.	2	Macedonia	3	Switzerland	4
Denmark	1	Netherlands	2		
France	2	Poland	1		

On the other hand, here's what you get in swedish :

Albanien	1	Liechtenstein	1	Slovakien	2
Belgien	1	Luxembourg	3	Slovenien	3
Bosnien	1	Macedonien	3	Spanien	3
Bulgarien	1	Nederländerna	2	Tjeckien	3
Danmark	1	Polen	1	Tyskland	4
Frankrike	2	Portugal	1	Ungern	1
Grekland	2	Rumänien	2	Österrike	5
Italien	1	Schweiz	3		
Kroatien	2	Serbien	3		

I claim that $\chi(G) = 4$. Since we've already exhibited a 4-coloring, we know that $\chi(G) \leq 4$. On the other hand, there are several reasons why it is clear that G cannot be colored with 3 colors, for example :

(a) France, Germany, Belgium and Luxembourg form a clique of size 4.
(b) Austria is at the centre of a 7-cycle formed by Germany, Czech Rep., Slovakia, Hungary, Slovenia, Italy and Switzerland. Being of odd length, this cycle will need at least 3 colors. Then a 4th will be needed for Austria.

(ii) For my Europe graph, $V = 25$, $E = 47$ and $R = 23$. Hence $V - E + R = 1$. For a correct graph of South America, $V = 13$, $E = 25$ and $R = 13$, so $V - E + R = 1$ again.

(iii) A football has 12 pentagons and 20 hexagons. Hence $R = 12+20 = 32$. Every node lies on exactly one pentagon, hence $V = 12 \cdot 5 = 60$. Every edge is shared between a pentagon and a hexagon, hence

$$E = \frac{1}{2}(12 \cdot 5 + 20 \cdot 6) = 90.$$

Thus $V - E + R = 2$ in this case.

(iv) First,

$$V - E + R = 1 \tag{1}$$

for any plane, connected graph. This is certainly true for any tree, since then $V = E + 1$ and $R = 0$. Given a plane, connected graph G, it has a spanning tree T. Then (1) holds for T. For every edge we add to T, we add exactly one region, since this edge may not cross any existing edge. Thus E increases by one and so does R, so (1) continues to hold.

A football can be considered as a plane graph provided we 'puncture' it by removing one region. Hence, 2 instead of 1.