TMA 055 : Diskret matematik

Tentamen 201003

Lösningar

F.1 $45 = 3^2 \cdot 5$ so $\phi(45) = \phi(3^2) \cdot \phi(5) = (3^2 - 3)(5 - 1) = 6 \cdot 4 = 24$. Hence, Euler's Theorem states that, if n is an integer relatively prime to 45, then

$$n^{24} \equiv 1 \pmod{45}.$$

Note that both 2 and 7 are relatively prime to 45. Hence (all congruences are modulo 45)

$$2^{76} = (2^{24})^3 \cdot 2^4 \equiv 1^3 \cdot 16 \equiv 16,$$

and

$$7^{98} = (7^{24})^4 \cdot 7^2 \equiv 1^4 \cdot 49 \equiv 1 \cdot 4 \equiv 4.$$

Thus,

$$(2^{76} + 7^{98})^3 \equiv (16 + 4)^3 = 20^3 = 400 \cdot 20 \equiv -5 \cdot 20 = -100 \equiv -10 \equiv 35.$$

So the answer is 35.

F.2 Let

$$G(x) = \sum_{n=0}^{\infty} u_n x^n$$

denote the generating function of the sequence (u_n). Let's rock !

$$\begin{aligned}(1 - 4x)G(x) &= u_0 + \sum_{n=1}^{\infty} (u_n - 4u_{n-1})x^n \\ &= 2 + \sum_{n=1}^{\infty} (2n + 1)x^n \\ &= 2 + 2 \cdot \sum_{n=1}^{\infty} nx^n + \sum_{n=1}^{\infty} x^n\end{aligned}$$

$$
\begin{aligned}
&= 2 + \frac{2x}{(1-x)^2} + \frac{x}{1-x} \\
&= \frac{2(1-x)^2 + 2x + x(1-x)}{(1-x)^2} \\
&= \frac{x^2 - x + 2}{(1-x)^2}.
\end{aligned}
$$

Thus

$$G(x) = \frac{x^2 - x + 2}{(1-x)^2(1-4x)}.$$

We seek a partial fraction decomposition

$$\frac{x^2 - x + 2}{(1-x)^2(1-4x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-4x}. \tag{1}$$

Clearing denominators, we have

$$x^2 - x + 2 = A(1-x)(1-4x) + B(1-4x) + C(1-x)^2.$$

Gathering coefficients, we get the following system of linear equations to solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

After the usual Gauβ elimination and back substitution (I omit the details), we get the solution

$$A = -\frac{5}{9}, \quad B = -\frac{2}{3}, \quad C = \frac{29}{9}.$$

Substituting into (1) and using the relations

$$
\begin{aligned}
\frac{1}{1-t} &= \sum_{n=0}^{\infty} t^n, \\
\frac{1}{(1-t)^2} &= \sum_{n=0}^{\infty} (n+1) t^n,
\end{aligned}
$$

we conclude that

$$F(x) = -\frac{5}{9} \sum_{n=0}^{\infty} x^n - \frac{2}{3} \sum_{n=0}^{\infty} (n+1) x^n + \frac{29}{9} \sum_{n=0}^{\infty} 4^n x^n.$$

Hence, it follows that

$$u_n = -\frac{5}{9} - \frac{2}{3}(n+1) + \frac{29}{9} \cdot 4^n.$$

F.3 *Step 0* : Since $3 \cdot 5 \equiv 1 \pmod 7$, the first congruence can be rewritten as

$$x \equiv 5 \pmod 7.$$

Step 1 : We compute the inverse of $13 \cdot 17$ modulo 7. Since $13 \cdot 17 \equiv (-1) \cdot 3 \equiv -3 \equiv 4 \pmod 7$, we seek a solution to

$$4a_1 \equiv 1 \pmod 7.$$

It's easy to spot that a solution is $a_1 = 2$.

Step 2 : Compute the inverse of $7 \cdot 17$ modulo 13. Since $7 \cdot 17 \equiv 7 \cdot 4 = 28 \equiv 2 \pmod{13}$, we must solve

$$2a_2 \equiv 1 \pmod{13}.$$

A solution is $a_2 = 7$.

Step 3 : Compute the inverse of $7 \cdot 13 = 91$ modulo 17. Since $91 \equiv 6 \pmod{17}$, we must solve

$$6a_3 \equiv 1 \pmod{17}.$$

A solution is $a_3 = 3$.

Step 4 : A solution to the three congruences is given by

$$\begin{aligned} x &= 5 \cdot a_1 \cdot (13 \cdot 17) + 2 \cdot a_2 \cdot (7 \cdot 17) + 4 \cdot a_3 \cdot (7 \cdot 13) \\ &= 5 \cdot 2 \cdot 13 \cdot 17 + 2 \cdot 7 \cdot 7 \cdot 17 + 4 \cdot 3 \cdot 7 \cdot 13 \\ &= 4968. \end{aligned}$$

Step 5 : The general solution is

$$x = 4968 + (7 \cdot 13 \cdot 17) \cdot n = 4968 + 1547n,$$

where n is an arbitrary integer. In particular, the smallest positive solution is $x = 327$, got by taking $n = -3$.)

F.4 Rewrite the equation as

$$y^3 = x^2 + 3x + 2 = (x+1)(x+2).$$

The HL is a product of two consecutive integers, which by necessity are relatively prime. Since their product is a perfect cube, FTA implies that each is itself a perfect cube. That is, there are integers z, w such that

$$x + 1 = z^3, \qquad x + 2 = w^3.$$

But then $w^3 - z^3 = 1$, which is only possible if either $w = 1$, $z = 0$ or $w = 0$, $z = -1$.

Thus, our equation has two solutions, namely

$$x = -1,\ y = 0, \quad \text{and} \quad x = -2,\ y = 0.$$

F.5 Reading from left to right and from top to bottom, let us label the vertices in the three columns as b, c, d (first column), e, f, g (second column), and h, i, j (third column).

(i) Clearly, $\chi(G_0) \geq 3$ since G_0 contains many triangles. In fact, $\chi(G_0) \geq 4$ since, for example, c lies at the centre of a 5-cycle formed by a, b, e, f, d. This cycle, being of odd length, will require at least 3 colors, and then a fourth will be needed for c.

On the other hand, if we apply the greedy algorithm to the nodes ordered alphabetically, then we get a 4-coloring, namely (the colors are $1, 2, 3, 4$)

a	1	g	1
b	2	h	2
c	3	i	3
d	2	j	2
e	1	z	1
f	4		

Hence $\chi(G_0) = 4$.

(ii) Apply Dijkstra's algorithm to build up the following tree

Step	Choice of edge	Labelling
1	ab	$b := 2$
2	ac	$c := 4$
3	ad	$d := 5$
4	be/ce	$e := 6$
5	cf	$f := 6$
6	dg	$g := 7$
7	eh	$h := 8$
8	ei/fi	$i := 8$
9	fj	$j := 8$
10	hz/jz	$z := 11$

Hence the shortest path from a to z has length 11. Depending on the choices you made in Steps 4 and 10, there are three possibilities for the shortest path, namely

$$\begin{aligned} &a \to b \to e \to h \to z, \\ &a \to c \to e \to h \to z, \\ &a \to c \to f \to j \to z.. \end{aligned}$$

F.6 Since $A_0 = C_0 = 1$ (the empty sequence works !), it suffices to prove that, for each $n > 0$,

$$A_n = \sum_{m=1}^{n} A_{m-1} A_{n-m}.$$

So fix $n > 0$. For each $m = 1, ..., n-1$, let $A(n, m)$ denote the number of sequences $a_1 \cdots a_n$ of length n for which

$$a_{m+1} = 0, \quad a_i > 0 \text{ for } i = 2, ..., m. \tag{2}$$

And let $A(n, n)$ denote the number of sequences of length n for which

$$a_i > 0 \;\text{ for all } i = 2, ..., n. \tag{3}$$

It is clear that

$$A_n = \sum_{m=1}^{n} A(n, m),$$

and hence it suffices to prove that

$$A(n, m) = A_{m-1}A_{n-m}, \quad \text{for } m = 1, ..., n. \tag{4}$$

First suppose $1 \leq m \leq n-1$ and let $a_1 \cdots a_n$ be one of the $A(n, m)$ sequences satisfying (2). The subsequence $a_{m+1} \cdots a_n$, of length $n-m$, satisfies exactly the same conditions as at the outest, hence there are A_{n-m} possibilities for it. Since $a_2 > 0$ and $a_2 \leq a_1 + 1$, we must have $a_2 = 1$. We also know that $a_i \geq 1$ for $i = 2, ..., m$. So if we let $b_i = a_i - 1$ for $i = 2, ..., m$, then the subsequence $b_2 \cdots b_m$, of length $m - 1$, satisfies exactly the same conditions as at the outset. Hence there are A_{m-1} possibilities for it, and hence in turn for the subsequence $a_2 \cdots a_m$. Finally, an application of the multiplication principle verifies (4).

There remains the case $m = n$. We must verify that $A(n, n) = A_{n-1}A_0 = A_{n-1}$. Let $a_1 \cdots a_n$ be one of the $A(n, n)$ sequences satisfying (3). Since $a_2 > 0$ and $a_2 \leq a_1 + 1$, we must have $a_2 = 1$. We also know that $a_i \geq 1$ for $i = 2, ..., n$. Hence, letting $b_i := a_i - 1$ for $i = 2, ..., n$, the sequence $b_2 \cdots b_n$, of length $n - 1$, satisfies exactly the same conditions as at the outset. Hence there are A_{n-1} possibilities for it, hence so also for the sequence $a_2 \cdots a_n$, v.s.v.