TMA 055 : Diskret matematik

Tentamen 100104

Lösningar

F.1 $60 = 2^3 \cdot 3^2$ so $\phi(72) = \phi(2^3) \cdot \phi(3^2) = (2^3 - 2^2)(3^2 - 3) = 4 \cdot 6 = 24$. Hence, Euler's Theorem states that, if n is an integer relatively prime to 72, then

$$n^{24} \equiv 1 \pmod{72}.$$

Note that both 7 and 5 are relatively prime to 72. Hence (all congruences are modulo 72)

$$7^{75} = (7^{24})^3 \cdot 7^3 \equiv 1^3 \cdot 49 \cdot 7 \equiv -23 \cdot 7 \equiv -161 \equiv -17,$$

and

$$5^{50} = (5^{24})^2 \cdot 5^2 \equiv 1^2 \cdot 25 \equiv 25.$$

Thus,

$$(7^{75} + 5^{50})^3 \equiv (-17 + 25)^3 = 8^3 = 64 \cdot 8 \equiv -8 \cdot 8 = -64 \equiv 8.$$

So the answer is 8.

F.2 Let

$$G(x) = \sum_{n=0}^{\infty} u_n x^n$$

denote the generating function of the sequence (u_n). Let's rock !

$$\begin{aligned}
(1 - 5x)G(x) &= u_0 + \sum_{n=1}^{\infty} (u_n - 5u_{n-1})x^n \\
&= 2 + \sum_{n=1}^{\infty} (4^n + 2)x^n \\
&= 2 + \sum_{n=1}^{\infty} (4x)^n + 2 \cdot \sum_{n=1}^{\infty} x^n \\
&= 2 + \frac{4x}{(1 - 4x)} + \frac{2x}{1 - x} \\
&= \frac{-4x^2 - 4x + 2}{(1 - 4x)(1 - x)}.
\end{aligned}$$

It follows that

$$G(x) = \frac{-4x^2 - 4x + 2}{(1-5x)(1-4x)(1-x)}.$$

We seek a partial fraction decomposition

$$\frac{-4x^2 - 4x + 2}{(1-5x)(1-4x)(1-x)} = \frac{A}{1-5x} + \frac{B}{1-4x} + \frac{C}{1-x}. \tag{1}$$

Clearing denominators, we have

$$-4x^2 - 4x + 2 = A(1-4x)(1-x) + B(1-5x)(1-x) + C(1-5x)(1-4x).$$

Gathering coefficients, we get the following system of linear equations to solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 5 & 6 & 9 \\ 4 & 5 & 20 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}.$$

After the usual Gauβ elimination and back substitution (I omit the details), we get the solution

$$A = \frac{13}{2}, \quad B = -4, \quad C = -\frac{1}{2}.$$

Substituting into (1) and using the relation

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n,$$

we conclude that

$$F(x) = \frac{13}{2} \sum_{n=0}^{\infty} 5^n x^n - 4 \sum_{n=0}^{\infty} 4^n x^n - \frac{1}{2} \sum_{n=0}^{\infty} x^n.$$

Hence, it follows that

$$u_n = \frac{13}{2} \cdot 5^n - 4^{n+1} - \frac{1}{2}.$$

F.3 *Step 0*: Since $2 \cdot 6 \equiv 1 \pmod{11}$, the first congruence can be rewritten as

$$x \equiv 7 \cdot 6 \pmod{11} \;\Rightarrow\; x \equiv 9 \pmod{11}.$$

Step 1 : We compute the inverse of $16 \cdot 23$ modulo 11. Since $16 \cdot 23 \equiv 5 \cdot 1 = 5 \pmod{11}$, we seek a solution to

$$5 \cdot a_1 \equiv 1 \pmod{11}.$$

So a solution is $a_1 = -2$.

Step 2 : Compute the inverse of $11 \cdot 23$ modulo 16. Since $11 \cdot 23 \equiv -5 \cdot 7 = -35 \equiv -3 \pmod{16}$, we must solve

$$-3 \cdot a_2 \equiv 1 \pmod{16}.$$

A solution is $a_2 = 5$.

Step 3 : Compute the inverse of $11 \cdot 16$ modulo 23. Since $11 \cdot 16 \equiv 11 \cdot -7 = -77 \equiv -8 \pmod{23}$, we must solve

$$-8 \cdot a_3 \equiv 1 \pmod{23}.$$

A solution is $a_3 = -3$.

Step 4 : A solution to the three congruences is given by

$$\begin{aligned} x &= 9 \cdot a_1 \cdot (16 \cdot 23) + 4 \cdot a_2 \cdot (11 \cdot 23) 1 \cdot a_3 \cdot (11 \cdot 16) \\ &= 9 \cdot (-2) \cdot 16 \cdot 23 + 4 \cdot 5 \cdot 11 \cdot 23 + 1 \cdot (-3) \cdot 11 \cdot 16 \\ &= -2092. \end{aligned}$$

Step 5 : The general solution is

$$x = -2092 + (11 \cdot 16 \cdot 23) \cdot n = -2092 + 4048n,$$

where n is an arbitrary integer. In particular, the smallest positive solution is $x = 1956$, got by taking $n = 1$.

F.4 We use the inclusion-exclusion principle. Let X denote the set of all permutations of $\{1, ..., 100\}$. Define three subsets A, B, C of X as follows :

$$\begin{aligned} A &= \{\pi \in X : \pi(1) \in \{10, 11\}\}, \\ B &= \{\pi \in X : \pi(2) \in \{20, 21\}\}, \\ C &= \{\pi \in X : \pi(3) \in \{30, 31\}\}. \end{aligned}$$

Then the number we're looking for is precisely $|X\backslash(A\cup B\cup C)|$. By the I-E principle, we have

$$|X\backslash(A\cup B\cup C)| = |X| - |A| - |B| - |C| \qquad (2)$$
$$+|A\cap B| + |A\cap C| + |B\cap C| - |A\cap B\cap C|.$$

Then we note that

$$|X| = 100! \qquad (3)$$

and

$$|A| = |B| = |C| = 2\cdot 99! \qquad (4)$$

and

$$|A\cap B| = |A\cap C| = |B\cap C| = 4\cdot 98! \qquad (5)$$

and

$$|A\cap B\cap C| = 8\cdot 97! \qquad (6)$$

Substituting (3), (4), (5) and (6) into (2), we obtain the final answer, namely

$$100! - 6\cdot 99! + 12\cdot 98! - 8\cdot 97!$$

F.5 (i) Use BFS, starting, say, from the vertex a, to build up the following sequence of edges in a MST :

$$\{a,d\},\{d,c\},\{c,f\},\{f,g\},\{f,j\},$$
$$\{j,k\},\{k,h\},\{h,e\},\{e,b\},\{j,i\}.$$

The total weight of this tree is $2+3+1+2+2+2+1+2+2+2 = 19$.

(ii) Apply Dijkstra's algorithm to build up the following tree

Step	Choice of edge	Labelling
1	ad	$d := 2$
2	ab	$b := 3$
3	ac	$c := 4$
4	be	$e := 5$
5	cf/df	$f := 5$
6	dg	$g := 6$
7	eh	$h := 7$
8	fj	$j := 7$
9	hk	$k := 8$

Hence the shortest path from a to k is the path $a \to b \to e \to h \to k$ and has length 8.

(iii) Not going to bother drawing them. There are 11 of them.

F.6 Recall the formula for a geometric sum

$$\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}.$$

Thus the repunit with $p-1$ digits is just

$$\sum_{k=0}^{p-2} 10^k = \frac{10^{p-1} - 1}{9}.$$

By Fermat's theorem, if p is a prime not dividing 10, i.e.: if p is neither 2 nor 5, then

$$10^{p-1} \equiv 1 \pmod{p} \Rightarrow p \mid 10^{p-1} - 1.$$

So, unless p divides 9, i.e.: unless $p = 3$, then p will also divide $(10^{p-1} - 1)/9$. Thus, the result holds for all primes p other than 2,3 and 5.