First practice exam

Solutions

1. FIRST SOLUTION (THE EASIER ONE) :

If you choose the number 1, then you can't choose 2, and so there are $\lambda(n-2, k-1)$ choices for the remaining $k-1$ numbers. If you don't choose 1, then you must choose all k of your numbers from $\{2, ..., n\}$, and hence there are $\lambda(n-1, k)$ ways to do this.

SECOND SOLUTION :

Using the method of Exercise 3 on Homework 1 you may prove the explicit formula

$$\lambda(n, k) = \binom{n-k+1}{k}. \tag{1}$$

For full marks, you must write out the argument which leads to (1). I refer you to my solution of Ex.3, Hwk. 1, and let you make the necessary notational adjustments (from 200 to n, and from 27 to k).

Having proven (1), the recurrence relation reads

$$\binom{n-k+1}{k} = \binom{n-k}{k} + \binom{n-k}{k-1}. \tag{2}$$

To verify (2), it is perhaps easiest to put $m := n-k+1$ so that (2) becomes

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1},$$

which is just Pascal's identity.

2. Let

$$F(x) = \sum_{n=0}^{\infty} u_n x^n$$

denote the generating function of the sequence (u_n). Let's rock !

$$\begin{aligned}(1-2x-3x^2)F(x) &= (u_0+u_1x)-2(u_0x)+\sum_{n=2}^{\infty}(u_n-2u_{n-1}-3u_{n-2})x^n \\ &= (1+x)-2(x)+\sum_{n=2}^{\infty}2^nx^n \\ &= 1-x+\frac{4x^2}{1-2x} \\ &= \frac{6x^2-3x+1}{1-2x}.\end{aligned}$$

Since

$$1-2x-3x^2=(1+x)(1-3x),$$

we conclude that

$$F(x)=\frac{6x^2-3x+1}{(1+x)(1-3x)(1-2x)}.$$

We seek a partial fraction decomposition

$$\frac{6x^2-3x+1}{(1+x)(1-3x)(1-2x)}=\frac{A}{1+x}+\frac{B}{1-3x}+\frac{C}{1-2x}. \tag{3}$$

Clearing denominators, we have

$$6x^2-3x+1=A(1-3x)(1-2x)+B(1+x)(1-2x)+C(1+x)(1-3x).$$

Gathering coefficients, we get the following system of linear equations to solve

$$\begin{pmatrix}1&1&1\\5&1&2\\6&-2&-3\end{pmatrix}\begin{pmatrix}A\\B\\C\end{pmatrix}=\begin{pmatrix}1\\3\\6\end{pmatrix}.$$

After the usual Gauβ elimination and back substitution (I omit the details), we get the solution

$$A=\frac{5}{6},\quad B=\frac{3}{2},\quad C=-\frac{4}{3}.$$

Substituting into (3) and using the relation

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n,$$

we conclude that

$$F(x) = \frac{5}{6}\sum_{n=0}^{\infty}(-1)^n x^n + \frac{3}{2}\sum_{n=0}^{\infty} 3^n x^n - \frac{4}{3}\sum_{n=0}^{\infty} 2^n x^n.$$

Hence, it follows that

$$u_n = \frac{5}{6}\cdot(-1)^n + \frac{3}{2}\cdot 3^n - \frac{4}{3}\cdot 2^n.$$

3. Det är lätt att se att $\text{SGD}(19, 43) = 1$ ty båda är primtal. Därför vet vi att det finns heltal x_0, y_0 så att

$$19x_0 + 43y_0 = 1. \tag{4}$$

Vi hittar först en lösning till (4) genom att köra Euklides algoritm fram och tillbaka. Framåt får vi

$$\begin{aligned} 43 &= 2\cdot 19 + 5,\\ 19 &= 3\cdot 5 + 4,\\ 5 &= 1\cdot 4 + 1,\\ 4 &= 4\cdot 4 + 0. \end{aligned}$$

Bakåt får vi då

$$\begin{aligned} 1 &= 5 - 4\\ &= 5 - (19 - 3\cdot 5)\\ &= 4\cdot 5 - 19\\ &= 4\cdot(43 - 2\cdot 19) - 19\\ &= 4\cdot 43 - 9\cdot 19. \end{aligned}$$

Därmed har vi hittat lösningen $x_0 = -9$, $y_0 = 4$. Genom att multiplicera dessa med 3000 så får vi en lösning (x_1, y_1) till

$$19x + 43y = 3000, \tag{5}$$

nämligen $x_1 = -27000$, $y_1 = 12000$. Den allmäna lösningen till (5) ges då av

$$x = -27000 + 43n, \tag{6}$$
$$y = 12000 - 19n \tag{7}$$

där n är ett godtyckligt heltal. Till sist är vi intresserade av lösningar för vilka både $x > 0$ och $y > 0$.

Å ena sidan

$$x > 0 \Leftrightarrow -27000 + 43n > 0 \Leftrightarrow 43n > 27000 \Leftrightarrow n > 627. \tag{8}$$

Å andra sidan

$$y > 0 \Leftrightarrow 12000 - 19n > 0 \Leftrightarrow 19n < 12000 \Leftrightarrow n < 632. \tag{9}$$

Från (8) och (9) får vi fyra möjligheter för n, nämligen $n = 628, 629, 630, 631$. Om vi sätter in dessa fyra värden i (6) och (7) så får vi fyra lösningar :

$x = 4, y = 68$ och $x = 47, y = 49$ och $x = 90, y = 30$, och $x = 133, y = 11$.

4. *Step 1* : We compute the inverse of $19 \cdot 29$ modulo 11. Since $19 \cdot 29 \equiv 8 \cdot 7 \equiv 56 \equiv 1 \pmod{11}$, we seek a solution to

$$1a_1 \equiv 1 \pmod{11}.$$

So, obviously, we can take $a_1 = 1$.

Step 2 : Compute the inverse of $11 \cdot 29$ modulo 19. Since $11 \cdot 29 \equiv 11 \cdot 10 \equiv 110 \equiv -4 \pmod{19}$, we must solve

$$-4a_2 \equiv 1 \pmod{19}.$$

A solution is $a_2 = -5$.

Step 3 : Compute the inverse of $11 \cdot 19 = 209$ modulo 29. Since $209 \equiv 6 \pmod{29}$, we seek a solution to

$$6a_3 \equiv 1 \pmod{29}.$$

A solution is given by $a_3 = 5$.

Step 4 : A solution to the three congruences is given by

$$\begin{aligned} x &= 1 \cdot a_1 \cdot (19 \cdot 29) + 3 \cdot a_2 \cdot (11 \cdot 29) + 5 \cdot a_3 \cdot (11 \cdot 19) \\ &= 1 \cdot 1 \cdot 19 \cdot 29 + 3 \cdot (-5) \cdot 11 \cdot 29 + 5 \cdot 5 \cdot 11 \cdot 19 \\ &= 991. \end{aligned}$$

Step 5 : The general solution is

$$x = 991 + (11 \cdot 19 \cdot 29) \cdot n = 991 + 6061n,$$

where n is an arbitrary integer. Hence, the smallest positive solution is indeed $x = 991$, got by taking $n = 0$.)

5 (i) The graph has Hamilton cycles, for example

$$A \to B \to E \to H \to K \to I \to J \to G \to F \to C \to D \to A.$$

(ii) Use BFS, starting, say, from the vertex A, to build up the following sequence of edges in a MST :

$$\{A, D\}, \{A, B\}, \{B, E\}, \{E, C\}, \{C, F\},$$
$$\{C, G\}, \{G, J\}, \{J, I\}, \{I, K\}, \{I, H\} \text{ (or } \{J, H\}).$$

The total weight of this tree is $1 + 2 + 1 + 5 + 1 + 2 + 1 + 1 + 2 + 7 = 23$.

(iii) Apply Dijkstra's algorithm to build up the following tree

Step	Choice of edge	Labelling
1	AD	$D := 1$
2	AB	$B := 2$
3	BE	$E := 3$
4	AC/DC	$C := 8$
5	CF	$F := 9$
6	DG/CG	$G := 10$
7	EH	$H := 11$
8	GJ	$J := 11$
9	JI/EI	$I := 12$
10	IK	$K := 14$

Hence the shortest path from A to K has length 14. Depending on the choices you made in Steps 4,6,8, there are four possibilities for the shortest path, namely

$$A \to B \to E \to I \to K,$$
$$A \to D \to G \to J \to I \to K,$$
$$A \to D \to C \to G \to J \to I \to K,$$
$$A \to C \to G \to J \to I \to K.$$

6. By Pythagoras' Theorem, you are looking for integer solutions to the equation

$$x^2 + y^2 = z^2.$$

Dividing by z^2, this is the same as looking for RATIONAL solutions to the equation

$$x^2 + y^2 = 1.$$

The rest of the solution will be written out along with the solutions to Homework 2, which will be posted on Monday.