**Third practice exam**

**Solutions**

**1.** $40 = 2^3 \cdot 5$ so $\phi(40) = \phi(2^3) \cdot \phi(5) = (2^3 - 2^2)(5-1) = 4 \cdot 4 = 16$. Hence, Euler's Theorem states that, if $n$ is an integer relatively prime to 40, then

$$n^{16} \equiv 1 \pmod{40}.$$

Note that both 3 and 7 are relatively prime to 40. Hence (all congruences are modulo 40)

$$3^{97} = (3^{16})^6 \cdot 3 \equiv 1^6 \cdot 3 \equiv 3,$$

and

$$7^{50} = (7^{16})^3 \cdot 7^2 \equiv 1^3 \cdot 49 \equiv 1 \cdot 9 \equiv 9.$$

Thus,

$$(3^{97} + 7^{50})^3 \equiv (3+9)^3 = 12^3 = 144 \cdot 12 \equiv 24 \cdot 12 = 288 \equiv 8.$$

So the answer is 8.

**2.** Let

$$G(x) = \sum_{n=0}^{\infty} u_n x^n$$

denote the generating function of the sequence $(u_n)$. Let's rock !

$$\begin{aligned}(1-3x)G(x) &= u_0 + \sum_{n=1}^{\infty}(u_n - 3u_{n-1})x^n \\ &= 1 + \sum_{n=1}^{\infty} nx^n \\ &= 1 + \frac{x}{(1-x)^2} \\ &= \frac{x^2 - x + 1}{(1-x)^2}.\end{aligned}$$

Thus

$$G(x) = \frac{x^2 - x + 1}{(1-x)^2(1-3x)}.$$

We seek a partial fraction decomposition

$$\frac{x^2 - x + 1}{(1-x)^2(1-3x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-3x}. \tag{1}$$

Clearing denominators, we have

$$x^2 - x + 1 = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2.$$

Gathering coefficients, we get the following system of linear equations to solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & 2 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

After the usual Gau$\beta$ elimination and back substitution (I omit the details), we get the solution

$$A = -\frac{1}{4}, \quad B = -\frac{1}{2}, \quad C = \frac{7}{4}.$$

Substituting into (1) and using the relations

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n,$$
$$\frac{1}{(1-t)^2} = \sum_{n=0}^{\infty} (n+1)t^n,$$

we conclude that

$$F(x) = -\frac{1}{4}\sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}(n+1)x^n + \frac{7}{4}\sum_{n=0}^{\infty} 3^n x^n.$$

Hence, it follows that

$$u_n = -\frac{1}{4} - \frac{n+1}{2} + \frac{7}{4}\cdot 3^n.$$

**3.** We use inclusion-exclusion. Let $X$ denote the set of all permutations of the numbers $1, ..., 100$. Define three subsets $A, B, C$ of $X$ as follows :

$$\begin{aligned} A &= \{\pi \in X : \pi \text{ places 43 and 59 side-by-side}\}, \\ B &= \{\pi \in X : \pi \text{ places 43 and 89 side-by-side}\}, \\ C &= \{\pi \in X : \pi \text{ places 59 and 89 side-by-side}\}. \end{aligned}$$

Then the number we want to compute is precisely $|X \backslash (A \cup B \cup C)|$. By the I-E principle, we have

$$\begin{aligned} |X \backslash (A \cup B \cup C)| &= |X| - |A| - |B| - |C| \\ &+ |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|. \end{aligned} \tag{2}$$

Firstly, $|X| = 100!$. Next, $|A| = 2 \cdot 99!$. For we may take the pair $\{43, 59\}$ as a single 'number'. We then have 99 'numbers' to permute, and also 2 choices for the ordering of the pair. Similarly, $|B| = |C| = 2 \cdot 99!$.

Now consider $A \cap B$ for example. This set consists of those permutations which place 43 next to 59 and 59 next to 89. Hence, the triple $\{43, 59, 89\}$ can first of all be considered as a single 'number', which leaves us with 98 'numbers' to permute freely. Within this triple, there are 2 possible orderings which satisfy our requirements, namely those that place 59 in the middle. Hence, we conclude that $|A \cap B| = 2 \cdot 98!$, and likewise for $|A \cap C|$ and $|B \cap C|$.

Finally, I claim that $|A \cap B \cap C| = 0$. For it is not possible to order 3 objects such that each is placed next to the other.

Substituting everything into (2) and adding/subtracting, the answer becomes $100! - 6 \cdot 99! + 6 \cdot 98!$.

**4.** After a little work we find the factorisation

$$n^4 + 6n^3 + 11n^2 + 6n = n(n+1)(n+2)(n+3) := p(n), \text{ say.}$$

Now $40 = 8 \cdot 5$ so a number is divisible by 40 if and only if it is divisible by both 8 and 5. I claim that $p(n)$ is divisible by 8 for every integer $n$. For $p(n)$ is the product of 4 consecutive integers. Two of these will always be even, and one of those two in turn will be divisible by 4.

Similarly, since 5 is prime, $p(n)$ is divisible by 5 if and only if at least one of its' factors is so. But among 4 consecutive numbers we always find a multiple of 5, except when the smallest of them is congruent to 1 modulo 5.

Hence, the final conclusion is that $p(n)$ is not divisible by 40 exactly when $n \equiv 1 \pmod 5$.

**5 (i)** Apply Dijkstra's algorithm to build up the following tree

| Step | Choice of edge | Labelling |
|---|---|---|
| 1 | $(s, b)$ | $b := 6$ |
| 2 | $(s, a)$ | $a := 7$ |
| 3 | $(s, c)$ | $c := 8$ |
| 4 | $(b, d)$ | $d := 10$ |
| 5 | $(a, f)$ | $f := 12$ |
| 6 | $(c, h)$ | $h := 12$ |
| 7 | $(d, g)$ | $g := 13$ |
| 8 | $(c, e)$ | $e := 14$ |
| 9 | $(h, t)$ | $t := 18$ |

Hence the shortest path from $s$ to $t$ is $s \to c \to h \to t$ and has length 18.

**(ii)** Suppose the contrary. Then $G$ is bipartite. Let the two 'sides' of $G$ contain $s$ and $t$ vertices respectively, where $s + t = n$. Then the maximum possible number of edges in $G$ is $st = s(n - s)$. Simple calculus shows that this quantity, as a function of $s$, attains a maximum of $n^2/4$ at $s = n/2$. Hence, if $G$ has no odd cycles it has at most $n^2/4$ edges, v.s.v.

**6.** We work modulo 5. Fermat's theorem states that, if $a$ is an integer not divisible by 5, then

$$a^4 \equiv 1 \pmod 5.$$

If $a$ IS divisible by 5, then naturally so is $a^4$, in other words $a^4 \equiv 0 \pmod 5$ in this case.

To sum up, $a^4 \equiv 0$ or $1 \pmod 5$ for all integers $a$. Hence, there are only 4 possibilities modulo 5 for the expression $x^4 - 3y^4$, namely

$$0 - 3 \cdot 0 \equiv 0,$$
$$1 - 3 \cdot 0 \equiv 1,$$
$$0 - 3 \cdot 1 \equiv 2,$$

$$1 - 3 \cdot 1 \equiv 3.$$

Hence, we see that the remainder of 4 modulo 5 is unattainable, and since this is precisely the remainder left by 19, we conclude that $x^4 - 3y^4$ can never equal 19, for any integers $x, y$, v.s.v.