TMA 055 : Diskret Matematik (E3)

Week 2

Demonstration problems for Wedenesday, Sept 10

1. Write the function

$$\frac{1}{(1+3x)^2}$$

as a power series.

2. Write each of the power series

$$\sum_{n=0}^{\infty} nx^n, \quad \sum_{n=0}^{\infty} n^2 x^n,$$

as rational functions, i.e.: in the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

3. Solve the recurrence relation

$$u_0 = 1, \quad u_1 = 4,$$
$$u_{n+2} = 4u_{n+1} - 4u_n,$$

twice, once using geneating functions and once not.

4. Using the method of generating functions (or otherwise), solve the recurrence relation

$$u_0 = 1, \quad u_1 = 2,$$
$$2u_{n+2} = 7u_{n+1} - 3u_n + n.$$

Demonstration problems for Friday, Sept 12

Recall the Generalised Binomial Theorem (GBT) :

Theorem *If x is a real number with $|x| < 1$ and z any complex number, then*

$$(1+x)^z = \sum_{k=0}^{\infty} \binom{z}{k} x^k,$$

where

$$\binom{z}{k} \overset{def}{=} \frac{z(z-1)\cdots(z-k+1)}{k!}.$$

1. Let n be a negative integer, say $n = -m$. Show that

$$\binom{n}{k} = (-1)^k \binom{m+k-1}{k}.$$

2. Repeat **Q.1** from Wednesday using the GBT. Also, write the rational function

$$\frac{1 - x + 2x^2}{1 + 4x + 6x^2 + 4x^3 + x^4}$$

as a power series.

3 (i) Explain why the Catalan numbers C_n satisfy the following alternative recurrence relation

$$C_0 = 1,$$

$$C_n = \sum_{k_1+\cdots+k_t=n} C_{k_1-1} C_{k_2-1} \cdots C_{k_t-1}, \qquad (1)$$

where the sum is taken over all ordered partitions of n into positive parts.
(ii) Write out the sum (1) in full for $n = 1, 2, 3, 4$. How many terms are in each sum ? Notice any pattern ? If so, prove a formula for the number of terms in the sum as a function of n. Compare this with the usual recurrence for the Catalan numbers.

Further practice problems

(this list will be constantly updated)

0 (11.1.7 in Biggs) Prove the identity

$$\binom{s-1}{0}+\binom{s}{1}+\cdots+\binom{s+n-2}{n-1}+\binom{s+n-1}{n}=\binom{s+n}{n},$$

where s, n are positive integers.

(Hint : If X is an $(s+n)$-element set and $Y = \{y_1, ..., y_n\}$ is a specific n-element subset of X, what is the number of n-element subsets of X for which y_r is the first member of Y not in the subset ?).

1. Let $(u_n)_{n=0}^{\infty}$ be a sequence of numbers. The *exponential generating function* of the sequence is the power series

$$E(x) = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!}.$$

(i) Suppose the sequence (u_n) satisfies the recurrence relation

$$au_{n+2} + bu_{n+1} + cu_n = f_n,$$

where (f_n) is some 'known' sequence. Show that in that case the e.g.f. satisfies the differential equation

$$aE''(x) + bE'(x) + cE(x) = f(x),$$

where

$$f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}.$$

(Note : The point of this exercise is to show those of you familiar with differential equations the explicit connection between linear recurrence relations with constant coefficients and differential equations of the same type).

(ii) Let d_n denote the number of derangements of $[n]$ as usual. In class we proved that

$$d_n = (n-1)(d_{n-1} + d_{n-2}), \qquad n \geq 3.$$

Using this or otherwise, prove that

$$d_n = nd_{n-1} + (-1)^n, \qquad n \geq 2. \tag{2}$$

(Note : With regard to the 'otherwise', I don't know of any more direct combinatorial proof of (1). It would be interesting if anyone could find one !).

Now use (1) to prove that the exponential generating function of the sequence d_n (we define $d_0 = 1$) is

$$E(x) = \frac{e^{-x}}{1-x}.$$

From this, recover the usual explicit formula for d_n (which we also derived in class).

2 (25.3.4 in Biggs) What is the coefficient of x^n in

$$\frac{1 + 2x + 2x^2}{1 - 3x + 3x^2 - x^3}$$

when written as a power series.

3. Let q_n denote the number of words of length n in the alphabet $\{a, b, c, d, e\}$ which contain no two consecutive a's. Find and solve a recurrence relation for q_n. Hence evaluate

$$\lim_{n \to \infty} \frac{q_{n+1}}{q_n}.$$

(Note : The numbers get a big ugly here, like in the formula for f_n).

4. Solve the recurrence relation

$$u_0 = 4, \quad u_1 = 1,$$
$$u_n = 4u_{n-1} + 5u_{n-2} + 3^n \ \ \forall\, n \geq 2.$$

5. Solve the recurrence relation

$$u_0 = 3, \quad u_1 = 4,$$
$$u_{n+2} = 5u_{n+1} - 6u_n + n.$$

6. Without using generating functions, can you guess the form of the general solution to the recurrence relation

$$8u_{n+3} = 12u_{n+2} - 6u_{n+1} + u_n \ ?$$

Now add in the initial conditions $u_0 = u_1 = u_2 = 1$ and solve the recurrence. Repeat using generating functions.

7 (see 25.3.3 in Biggs) Recall that the generating function for the Fibonacci numbers (f_n) satisfies

$$G(x) = \frac{1}{1 - x - x^2}.$$

Using this fact, prove the following

(i) f_n is the number of ordered partitions of n into parts each of which is either one or two.

(ii) we have the formula

$$f_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-r}{r},$$

where r is the greatest integer such that $r \leq n - r$.

Now try to reprove these two facts, but without explicit use of the generating function.

8. Let P_n be any diagonal path in the plane from $(0, 0)$ to $(2n, 0)$. Recall that a Dyck path is such a path which moreover doesn't go below the x-axis. Let $A(P_n)$ denote the area under the path, where areas below the x-axis are counted as negative.

(i) Show that, for any path P_n, $A(P_n)$ is an integer and $A(P_n) - n$ an even integer.

For each integer k such that $k - n$ is even, let $N(n, k)$ denote the number of diagonal paths P_n of length $2n$ such that $A(P_n) = k$.

(ii) Show that $N(n, k) = N(n, -k)$.

(iii) Compute the smallest positive integer k, as a function of n, for which $k - n$ is even and $N(n, k) = 0$. Show that $N(n, k) > 0$ for every smaller positive integer k such that $k - n$ is even.

(iv) List the numbers $N(n, k)$ for $n = 1, 2, 3, 4, 5$.

(Note : It is known that the function $N(n, k)$, as a function of k for any fixed n, is *unimodal*, i.e.: if we only consider those k for which $k - n$ is even, then $N(n, k)$ decreases monotonically as $|k|$ increases. There are several known proofs of this fact, none of them easy !!).