Week 2 practice problems : Solutions

1 (i) The point is to show that

$$E'(x) = \sum_{n=0}^{\infty} u_{n+1} \frac{x^n}{n!}, \tag{1}$$

from which it follows in turn that

$$E''(x) = \sum_{n=0}^{\infty} u_{n+2} \frac{x^n}{n!}. \tag{2}$$

(1) and (2) immediately imply that

$$aE''(x) + bE'(x) + cE(x) = f(x),$$

as required. To prove (1), just compute :

$$\begin{aligned} E'(x) = \frac{d}{dx} \sum_{n=0}^{\infty} u_n \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \frac{d}{dx} \left(u_n \frac{x^n}{n!} \right), \\ &= \sum_{n=1}^{\infty} u_n \cdot \frac{nx^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} u_n \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} u_{n+1} \frac{x^n}{n!}, \quad \text{v.s.v..} \end{aligned}$$

(ii) Since $d_1 = 0$ and $d_2 = 1$ we may verify directly for $k = 2$ that

$$d_k = d_{k-1} + (-1)^k. \tag{3}$$

We now proceed by induction on k. Suppose (3) holds for $k = n - 1$, i.e.: that

$$d_{n-1} = (n-1)d_{n-2} + (-1)^{n-1}.$$

We rewrite this as

$$d_{n-2} = \frac{1}{n-1} \left[d_{n-1} - (-1)^{n-1} \right] = \frac{1}{n-1} \left[d_{n-1} + (-1)^n \right]. \tag{4}$$

We know that

$$d_n = (n-1)(d_{n-1} + d_{n-2}).$$

Substituting (4) into this we find that

$$d_n = (n-1)\left[d_{n-1} + \frac{1}{n-1}\left(d_{n-1} + (-1)^n\right)\right] = nd_{n-1} + (-1)^n, \quad \text{v.s.v..}$$

Now let $E(x)$ be the exp. generating function of the sequence (d_n). In the following computation, note that $d_0 = 1$, $d_1 = 0$ and $d_2 = 1$. We have

$$\begin{aligned} xE(x) &= \sum_{n=0}^{\infty} d_0 \frac{x^{n+1}}{n!} \\ &= x + \sum_{n=2}^{\infty} d_n \frac{x^{n+1}}{n!} \\ &= x + \sum_{n=3}^{\infty} nd_{n-1} \frac{x^n}{n!} \\ &= x + \sum_{n=3}^{\infty} (d_n - (-1)^n) \frac{x^n}{n!} \\ &= x + \sum_{n=3}^{\infty} d_n \frac{x^n}{n!} - \sum_{n=3}^{\infty} \frac{(-x)^n}{n!} \\ &= x + \left(E(x) - 1 - \frac{x^2}{2!}\right) - \left(e^{-x} - 1 - (-x) - \frac{(-x)^2}{2!}\right) \\ &= E(x) - e^{-x}. \end{aligned}$$

That is,

$$xE(x) = E(x) - e^{-x} \Rightarrow E(x) = \frac{e^{-x}}{1-x}, \quad \text{v.s.v..}$$

So now we have

$$\begin{aligned} E(x) &= e^{-x}(1-x)^{-1} \\ &= \left(\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}\right) \cdot \left(\sum_{l=0}^{\infty} x^l\right). \end{aligned}$$

The coefficient of x^n in this expression is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

On the other hand, this coefficient must be $\frac{d_n}{n!}$. Hence it follows that

$$\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!},$$

which is the formula we were supposed to recover.

3. Consider words of length $n > 1$. If the first letter is an a, then the second must be b, c, d or e (i.e.: 4 choices), and then there are q_{n-2} choices for the remaining $n-2$ letters. Hence, there are $4q_{n-2}$ words of this type.

If the first letter is not an a, then it is one of b, c, d or e (i.e.: 4 choices), and then there are q_{n-1} choices for the remaining $n-1$ letters. Thus there are $4q_{n-1}$ words of this type.

From the above discussion, we deduce the recurrence relation

$$q_n = 4q_{n-1} + 4q_{n-2}, \quad \text{for all } n \geq 2. \tag{5}$$

By inspection, we have the initial conditions

$$q_0 = 1, \quad q_1 = 5. \tag{6}$$

The auxiliary quadratic equation is

$$x^2 - 4x - 4 = 0,$$

which has the two real roots

$$x = 2(1 \pm \sqrt{2}).$$

Hence the general solution to (5) is

$$q_n = 2^n \left[C_1 \left(1 + \sqrt{2}\right)^n + C_2 \left(1 - \sqrt{2}\right)^n \right].$$

The initial conditions (6) yield two equations for C_1 and C_2 which, after a little algebra, are found to be

$$C_1 = \frac{3 + 2\sqrt{2}}{4\sqrt{2}}, \quad C_2 = \frac{-3 + 2\sqrt{2}}{4\sqrt{2}}.$$

Thus we conclude that

$$q_n = 2^n \left[\left(\frac{3 + 2\sqrt{2}}{4\sqrt{2}} \right) \left(1 + \sqrt{2}\right)^n + \left(\frac{-3 + 2\sqrt{2}}{4\sqrt{2}} \right) \left(1 - \sqrt{2}\right)^n \right].$$

5. Let

$$G(x) = \sum_{n=0}^{\infty} u_n x^n$$

denote the generating function of the sequence (u_n). Let's rock !

$$\begin{aligned}
(1-5x+6x^2)F(x) &= (u_0 + u_1 x) - 5(u_0 x) + \sum_{n=2}^{\infty}(u_n - 5u_{n-1} + 6u_{n-2})x^n \\
&= (3+4x) - 5(3x) + \sum_{n=2}^{\infty}(n-2)x^n \\
&= 3 - 11x + \sum_{n=0}^{\infty} n x^{n+2} \\
&= 3 - 11x + x^2 \cdot \sum_{n=0}^{\infty} n x^n \\
&= 3 - 11x + x^2 \cdot \sum_{n=1}^{\infty} n x^n \\
&= 3 - 11x + x^2 \cdot x \cdot \sum_{n=1}^{\infty} n x^{n-1} \\
&= 3 - 11x + \frac{x^3}{(1-x)^2} \\
&= \frac{-10x^3 + 25x^2 - 17x + 3}{(1-x)^2}.
\end{aligned}$$

Since

$$1 - 5x + 6x^2 = (1-2x)(1-3x),$$

we conclude that

$$G(x) = \frac{-10x^3 + 25x^2 - 17x + 3}{(1-x)^2(1-2x)(1-3x)}.$$

We seek a partial fraction decomposition

$$\frac{-10x^3 + 25x^2 - 17x + 3}{(1-x)^2(1-2x)(1-3x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-2x} + \frac{D}{1-3x}. \tag{7}$$

Clearing denominators, we have

$$\begin{aligned}-10x^3+25x^2-17x+3 &= A(1-x)(1-2x)(1-3x)+B(1-2x)(1-3x)\\ &\quad +C(1-x)^2(1-3x)+D(1-x)^2(1-2x).\end{aligned}$$

Gathering coefficients, we get the following system of linear equations to solve

$$\begin{pmatrix} 1 & 1 & 1 & 1\\ 6 & 5 & 5 & 4\\ 11 & 6 & 7 & 5\\ 6 & 0 & 3 & 2\end{pmatrix}\begin{pmatrix} A\\ B\\ C\\ D\end{pmatrix}=\begin{pmatrix} 3\\ 17\\ 25\\ 10\end{pmatrix}.$$

After the usual Gauβ elimination and back substitution (I omit the details), we get the solution

$$A=\frac{1}{4},\quad B=\frac{1}{2},\quad C=4,\quad D=-\frac{7}{4}.$$

Substituting into (7) and using the relations

$$\begin{aligned}\frac{1}{1-t}&=\sum_{n=0}^{\infty}t^n,\\ \frac{1}{(1-t)^2}&=\sum_{n=0}^{\infty}(n+1)t^n,\end{aligned}$$

we conclude that

$$F(x)=\frac{1}{4}\sum_{n=0}^{\infty}x^n+\frac{1}{2}\sum_{n=0}^{\infty}(n+1)x^n+4\sum_{n=0}^{\infty}2^nx^n-\frac{7}{4}\sum_{n=0}^{\infty}3^nx^n.$$

Hence, it follows that

$$\begin{aligned}u_n&=\frac{1}{4}+\frac{1}{2}(n+1)+4\cdot 2^n-\frac{7}{4}\cdot 3^n\\ &=\frac{3}{4}+\frac{n}{2}+4\cdot 2^n-\frac{7}{4}\cdot 3^n.\end{aligned}$$

7 (i) f_n is the coeffiecient of x^n in the power-series expansion of $G(x)$. But

$$G(x)=\frac{1}{1-x-x^2}=\frac{1}{1-(x+x^2)}=\sum_{k=0}^{\infty}(x+x^2)^k.$$

It is now 'clear' (!!) that the coefficient of x^n in this series equals the number of ways of writing n as an ordered sum of 1's and 2's.

(ii) On the other hand, x^n appears in the binomial expansion of $(x+x^2)^k$ if and only if $n/2 \leq k \leq n$, in which case the coefficient for x^n is $\binom{k}{2k-n}$. Hence,

$$f_n = \sum_{k=\lceil n/2 \rceil}^{n} \binom{k}{2k-n}.$$

If we make the change of variables $r := n - k$, then

$$f_n = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{n-2r} = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r}, \quad \text{v.s.v..}$$

Now let's try to reprove **(i)** without using $G(x)$. Let g_n denote the number of ways to write n as an ordered sum of 1's and 2's. It suffices to show that the g_n satisfy the same recurrence relation as the f_n, namely that

$$g_0 = g_1 = 1, \tag{8}$$
$$g_n = g_{n-1} + g_{n-2}, \quad \forall\, n \geq 2. \tag{9}$$

The initial conditions (8) are verified by inspection. To verify (9), we argue as follows : if we write n as an ordered sum of 1's and 2's and the last part is a 2, then the remaining parts sum to $n-2$, and hence there are g_{n-2} possibilities for them. Otherwise, if the last part is a 1, then the remaining parts sum to $n-1$ and there are g_{n-1} possibilities for them.

Finally, let us reprove **(ii)** without using $G(x)$. The formula is correct for $n = 0$, by inspection. For $n > 0$, we use the description of f_n as the number of $(n-1)$-digit binary words not containing any two consecutive zeroes. The number of digits in any such word is at most $\lceil (n-1)/2 \rceil = \lfloor n/2 \rfloor$. I claim that, for each $r = 0, ..., \lfloor n/2 \rfloor$, there are $\binom{n-r}{r}$ such words containing exactly r zeroes. Suppose the zeroes are in positions

$$1 \leq x_1 < x_2 < \cdots < x_r \leq n-1.$$

Define new variables $y_1, ..., y_{r+1}$ by

$$\begin{aligned} y_1 &:= x_1 - 1, \\ y_k := x_k - x_{k-1} - 2, &\quad \text{for } k = 2, ..., r, \\ y_{r+1} &:= (n-1) - x_r. \end{aligned}$$

The requirement that no two zeroes are consecutive then translates into the requirement that all y_i be non-negative integers. We may compute that

$$y_1 + \cdots + y_{r+1} = n - 2r. \tag{10}$$

By the morse-code method, we know that the number of solutions in non-negative integers to (10) is

$$\binom{n - 2r + (r+1) - 1}{(r+1) - 1} = \binom{n-r}{r}, \quad \text{v.s.v.}$$