TMA 055 : Diskret Matematik (E3)

Week 3

Demonstration problems for Wedenesday, Sept 17

1 (11.4.2 in Biggs) Find the number of ways of arranging the letters A,E,M,O,U,Y in a sequence in such a way that the words ME and YOU do not occur.

2. A function f from a set X to a set Y is said to be *surjective* if, for every $y \in Y$, there exists at least one $x \in X$ such that $f(x) = y$.

(a) Find the number of surjective functions from the set $\{1, 2, 3, 4, 5\}$ to the set $\{1, 2, 3\}$.

(b) More generally, let $m > n$ be positive integers and let $A_{m,n}$ denote the number of surjective functions from the set $\{1, 2, ..., m\}$ to the set $\{1, 2, ..., n\}$. Explain why

$$A_{m,n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Demonstration problems for Friday, Sept 19

Definitions : A relation $\mathcal{R}$ on a set X is said to be *antisymmetric* if

$$x\mathcal{R}y \text{ and } y\mathcal{R}x \Rightarrow x = y.$$

A relation which is reflexive, transitive and antisymmetric is called a *partial order*.

1. Which of the following relations are reflexive/symmetic/antisymmetric/transitive/equivalence relations/partial orders ? In the case of an equivalence relation, describe the equivalence classes.

(a) X is the set of all lines in the plane.

$$\mathcal{R} = \{(L, M) \in X \times X : L \perp M\}.$$

(b) $X = \mathbf{Z}$, $\mathcal{R} = \{(a, b) : |a - b| < 3\}$.

(c) $X = \mathbf{Z}$, $\mathcal{R} = \{(a, b) : a \leq b\}$.

(d) $X = \mathbf{Z}$, $\mathcal{R} = \{(a, b) : a + b \text{ is even}\}$.

2. For a positive integer n let $d(n)$ denote the numbr of positive integers that divide n, including 1 and n itself. Let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

be the prime decomposition of n. Find a formula for $d(n)$. Hence compute $d(3000)$.

3. Let p be a prime and $1 \leq i \leq p - 1$. Prove that the binomial coefficient $\begin{pmatrix} p \\ i \end{pmatrix}$ is divisible by p. (we will make use of this result later in the lectures).

4. Show that there are no integer solution to the equation

$$x^2 - 2y^2 = 5.$$

Further practice problems

(this list will be constantly updated)

1. Without listing them all, compute the number of positive integers less than or equal to 10,000 which have no common factor with 60.

2. Compute, in terms of binomial coefficients, the number of solutions to the equation

$$a + b + c + d + e = 127,$$

where a, b, c, d, e are non-negative integers such that $a \leq 12$, $b \leq 10$ and $c \leq 15$.

3 (recommended as practice for homework 2) A permutation $a_1, a_2, ..., a_n$ of the integers $1, 2, ..., n$ is said to be $1 - 3 - 2$ *avoiding* if there does not exist any three integers i, j, k such that

$$1 \leq i < j < k \leq n$$

and

$$a_i < a_j > a_k > a_i.$$

Write out all 1-3-2 avoiding permutations of $\{1, 2, ..., n\}$ for $n = 1, 2, 3, 4$. Let A_n denote the number of such permutations. Show that $A_n = C_n$, the n:th Catalan number.

(Hint : By considering the position of n in a permutation, show that the A_n satisfy the same recurrence relation as the C_n.)