Week 3 practice problems : Solutions

1. We use inclusion-exclusion. Let $X = \{1, 2, ..., 10000\}$. Define three subsets A, B, C of X as follows :

$$\begin{aligned} A &= \{n \in X : n \text{ is divisible by } 2\}, \\ B &= \{n \in X : n \text{ is divisible by } 3\}, \\ C &= \{n \in X : n \text{ is divisible by } 5\}. \end{aligned}$$

Since 2,3,5 are the different primes dividing 60, the number of integers in the set X which are relatively prime to 60 is precisely $|X \backslash (A \cup B \cup C)|$. By the I-E principle, we have

$$\begin{aligned} |X \backslash (A \cup B \cup C)| &= |X| - |A| - |B| - |C| \\ +|A \cap B| + |A \cap C| &+ |B \cap C| - |A \cap B \cap C|. \end{aligned} \tag{1}$$

Note that

$$\begin{aligned} A \cap B &= \{n \in X : n \text{ is divisible by } 6\}, \\ A \cap C &= \{n \in X : n \text{ is divisible by } 10\}, \\ B \cap C &= \{n \in X : n \text{ is divisible by } 15\}, \\ A \cap B \cap C &= \{n \in X : n \text{ is divisible by } 30\}. \end{aligned}$$

Now all terms on the HL of (1) are simple to compute : we have

$$\begin{aligned} |X| &= 10,000 \\ |A| &= 5,000 \\ |B| &= 3,333 \\ |C| &= 2,000 \\ |A \cap B| &= 1,666 \\ |A \cap C| &= 1,000 \\ |B \cap C| &= 666 \\ |A \cap B \cap C| &= 333. \end{aligned}$$

Substituting everything into (1) and adding/subtracting, the answer becomes 2666.

3. Since $A_0 = C_0 = 1$, it suffices to show that the A_n satisfy the same recurrence relation as the Catalan numbers, namely that

$$A_n = \sum_{m=1}^{n} A_{m-1}A_{n-m}, \quad \text{for all } n \geq 1. \tag{2}$$

Let $a_1 \cdots a_n$ be a $1-3-2$ avoiding permutation of $1, ..., n$, and suppose $a_m = n$. Then each of the numbers a_i for $1 \leq i < m$ must be greater than each of the numbers a_i for $i > m$. In other words, $a_1 \cdots a_{m-1}$ is a permutation of the numbers $n-m+1, n-m+2, ..., n-1$ and $a_{m+1} \cdots a_n$ is a permutation of the numbers $1, 2, ..., n-m$. Each of these permutations must also be $1-3-2$ avoiding, hence there are A_{m-1} and A_{n-m} possibilities for them, respectively. By MP, we conclude that there are $A_{m-1}A_{n-m}$ $1-3-2$ avoiding permutations of $1, ..., n$ for which n is placed in the m:th position. Summing over m from 1 to n, we have verified (2).