TMA 055 : Diskret Matematik (E3)

Week 4

Demonstration problems for Tuesday, Sept 23

1. Prove that $\sqrt{2}$ is an irrational number.

2. Let p be a prime. Consider the function $f : \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ given by $x \mapsto x^2$. Calculate the size of the image of f.

3. Find the general solution to the Diophantine equation

$$23x + 41y = 2000$$

and find all solutions for which $x > 0$, $y > 0$.

4. Compute the remainder left by

$$(3^{122} + 7^{36})^{44}$$

after division by 13.

Demonstration problems for Thursday, Sept. 25

1. Find all solutions (if any exist) in $\mathbf{Z}/11\mathbf{Z}$ to each of the congruences

$$x^2 + 3x + 7 \equiv 0 \pmod{11},$$
$$x^2 + 3x + 8 \equiv 0 \pmod{11}.$$

2. With the help of Fermats theorem, evaluate the remainder when

$$(7^{94} + 2^{43})^{68}$$

is divided by 23.

3. Let n be a given positive integer. Suppose you are also told the value of $\phi(n)$ and that n is a product of two distinct primes p and q. How would you go about finding p and q ?

4. FInd all solutions $x \in \mathbf{Z}$ to the congruences

$$x \equiv 1 \pmod{11},$$
$$x \equiv 2 \pmod{17},$$
$$x \equiv 3 \pmod{23}.$$

Further practice problems

(this list will be constantly updated)

0. For which integers n is

$$n^3 + 3n^2 + 2n$$

divisible by 12 ?

(Hint : Factorise everything in sight).

1. Prove the following theorem :

Let

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with integer coefficients. Suppose the polynomial has a rational root p/q, where $SGD(p,q) = 1$. Then we must have

$$p | a_0, \qquad q | a_n.$$

Hence write down a cubic polynomial with integer coefficients, which has no rational root.

2. Find the general solution to the Diophantine equation

$$35x + 15y + 21z = 1.$$

(Hint : First solve $35x + 15y = 5$).

3. Formulate a method for testing whether a number is divisible by 9 which only involves the digits of the decimal expansion of the number. Formulate a similar test for divisibility by 11.

4. Show that $\sqrt{10}$ is an irrational number. Hence show that

$$\sqrt{5} + \sqrt{2}$$

is an irrational number.

(Hint : Suppose the contrary. Square it and use irrationality of $\sqrt{10}$).

5. By considering the number

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$

prove that there exists some pair a, b of irrational numbers such that a^b is rational.

6. Let's say the universe will end in exactly 10^{15} earth years from today, Tuesday, Sept. 23. On what weekday will the universe end ?

(Assume that every year contains 365 days, except for a leap year every fourth year. Ignore the more complicated reality !!!).

7. Let $f : \mathbf{N} \to \mathbf{N}$ be a function with the property that

$$f(mn) = f(m)f(n), \quad \text{whenever } \mathrm{SGD}(m,n) = 1. \tag{1}$$

Now consider a new function $g : \mathbf{N} \to \mathbf{N}$ defined by

$$g(n) \stackrel{\text{def}}{=} \sum_{d|n} f(d),$$

where the sum is taken over all positive integers d which divide n, including 1 and n itself.

Show that the function g also satisfies (1). Hence, or otherwise, prove that, for every integer $n > 0$,

$$\sum_{d|n} \phi(d) = n.$$