Week 4 practice problems : Solutions

0. Factorise

$$n^3 + 3n^2 + 2n = n(n^2 + 3n + 2) = n(n+1)(n+2).$$

As a product of 3 consecutive numbers, this is always divisible by 3. One checks readily that it is divisible by 4 for $n = 0, 2, 3$ but not for $n = 1$.

Hence, the conclusion is that $n^3 + 3n^2 + 2n$ is divisible by 12 if and only if $n \not\equiv 1 \pmod 4$.

1 (i) Suppose $x = p/q$ is a rational root, where $\text{SGD}(p, q) = 1$. Then

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1} \left(\frac{p}{q}\right)^{n-1} + \cdots + a_1 \left(\frac{p}{q}\right) + a_0 = 0.$$

If we multiply through by q^n, then all denominators are cleared and we have that

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0. \tag{1}$$

The HL of (1), being zero, is obviously divisible by q. But some power of q appears in every term on the VL except the first. Hence, it must be that $q | a_n p^n$, and since $\text{SGD}(p, q) = 1$, FTA implies that $q | a_n$.

SImilarly, the HL of (1) is divisible by p. Every term on the VL includes some power of p except the last. Hence it must be that $p | a_0 q^n$, and sicne $\text{SGD}(p, q) = 1$, FTA implies that $p | a_0$, v.s.v.

(ii) There's so much to choose from, but consider for example the polynomial $x^3 + x^2 + 3x + 1$. Suppose p/q were a rational root, where $\text{SGD}(p, q) = 1$. The conditions $p | a_0$, $q | a_n$ now say that $p | 1$ and $q | 1$. Since p, q are integers, the only possibility is that both p and q are ± 1 and hence $p/q = \pm 1$ also. So one just needs to check that neither $x = \pm 1$ is a root of our polynomial.

2. Noting that $5 = \text{SGD}(35, 15)$, we first solve

$$35a + 15b = 5.$$

One solution is immediately obvious, namely

$$a_0 = 1, \quad b_0 = -2.$$

Hence the general solution is

$$a = 1 + 3m, \quad b = -2 - 7m, \qquad m \in \mathbf{Z}.$$

And, for any fixed $c \in \mathbf{Z}$, the general solution to

$$35a + 15b = 5c$$

is

$$a = c + 3m, \quad b = -2c - 7m, \qquad m \in \mathbf{Z}. \tag{2}$$

Next we solve

$$5c + 21z = 1.$$

A solution is also pretty obvious, namely

$$c_0 = -4, \quad z_0 = 1.$$

Hence the general solution is

$$c = -4 + 21n, \quad z = 1 - 5n, \qquad n \in \mathbf{Z}. \tag{3}$$

Combining (2) and (3), we find that the general solution to

$$35x + 15y + 21z = 1$$

is

$$\begin{aligned} x &= (-4 + 21n) + 3m, \\ y &= -2(-4 + 21n) - 7m, \\ z &= 1 - 5n, \end{aligned}$$

where m, n are arbitrary integers.

3. Let $n = x_k x_{k-1} \cdots x_1 x_0$ be the base-10 expansion of the number n, i.e.: each $x_i \in \{0, ..., 9\}$, $x_k > 0$ and

$$n = \sum_{i=0}^{k} x_i 10^i.$$

Now $10 \equiv 1 \pmod 9$ and so the same is true for every power of 10. Hence

$$n = \sum x_i 10^i \equiv \sum x_i \cdot 1 = \sum x_i \pmod{10}.$$

In other words, a number is divisible by 9 if and only if the same is true of the sum of its' base-10 digits.

We get a similarly simple rule for divisibility by 11. This time $10 \equiv -1 \pmod{11}$ and hence $10^i \equiv (-1)^i \pmod{11}$. Thus

$$n = \sum x_i 10^i \equiv \sum x_i (-1)^i \pmod{11}.$$

In other words, a number is divisible by 11 if and only if the same is true of the alternating sum of its' base-10 digits, reading from right to left.

4 (i) On the contrary, suppose that $\sqrt{10}$ were rational, say

$$\sqrt{10} = \frac{p}{q}, \quad \text{where } \mathrm{SGD}(p,q) = 1.$$

Sqyuaring both sides we get that $10 = p^2/q^2$ and hence

$$2 \cdot 5 \cdot q^2 = p^2. \tag{4}$$

The VL of (4) is divisible by 2, hence $2|p^2$, and so $2|p$. But then $4|p^2$. So dividing both sides of (4) by 2 we get

$$5q^2 = 2\left(\frac{p^2}{4}\right) \tag{5}$$

and the HL of (5) is still an even integer. Hence $2|\mathrm{VL} \Rightarrow 2|q^2 \Rightarrow 2|q$. So both p and q are even integers, contradicting the assumption that $\mathrm{SGD}(p,q) = 1$.

(ii) On the contrary, suppose that $\sqrt{5} + \sqrt{2}$ were rational, say equal to p/q. Then

$$\left(\frac{p}{q}\right)^2 = (\sqrt{5} + \sqrt{2})^2 = 5 + 2 + 2\sqrt{10},$$

and hence

$$\sqrt{10} = \frac{1}{2}\left(\frac{p^2}{q^2} - 7\right). \tag{6}$$

Eq. (6) says in particular that $\sqrt{10}$ is a rational number, which contradicts part **(i)** of this exercise.

5. We have that

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}\cdot\sqrt{2}} = (\sqrt{2})^2 = 2,$$

is a rational number. Also, we know that $\sqrt{2}$ is irrational. We don't know if $\sqrt{2}^{\sqrt{2}}$ is rational or not, but it doesn't matter :

For if $\sqrt{2}^{\sqrt{2}}$ is rational, take $a = b = \sqrt{2}$. If it isn't, take $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

In either case, a and b are two irrational numbers such that a^b is rational.

6. I'm writing these solutions on Saturday, Oct. 4, 2003. Note that $365 \equiv 1 \pmod 7$. Since every fourth year contains an extra day, we can say that every group of 4 years, from today onwards, shifts the day on which Oct. 4 falls forward 5 days or, which is the same thing, shifts it back 2 days - so Oct.4, 2007 will be a Thursday, for example.

Anyway, the point is that the day on which Oct. 4 falls in 10^{15} years will be shifted from a Saturday by x days where

$$x \equiv \frac{10^{15}}{4} \times (-2) = 25 \times 10^{13} \times -2 \pmod 7.$$

So it remains to compute this big number modulo 7. First, $25 \cdot (-2) \equiv 4 \cdot (-2) \equiv -8 \equiv -1$. Fermat's theorem implies that $10^6 \equiv 1$. Hence

$$25 \cdot (-2) \cdot 10^{13} \equiv (-1) \cdot (10^6)^2 \cdot 10 \equiv (-1) \cdot 1^2 \cdot 10 \equiv -3 \equiv 4 \pmod 7.$$

Hence, the universe will end on a Wednesday.

7 (i) This is a consequence of FTA. More precisely, FTA implies that if m, n are two relatively prime integers, then we have a 1-1 correspondence

$$\{\text{divisors of } m\} \times \{\text{divisors of } n\} \leftrightarrow \{\text{divisors of } mn\}$$

given by

$$(d_1, d_2) \leftrightarrow d_1 d_2. \tag{7}$$

Now let's show that

$$g(mn) = g(m)g(n), \quad \text{whenever SGD}(m, n) = 1. \tag{8}$$

By definition,

$$g(mn) = \sum_{d|mn} f(d).$$

By the corrspondence (7) and the fact that f already satisfies (8), we have

$$\sum_{d|mn} f(d) = \sum_{d_1 d_2 \text{ s.t. } d_1|m,\, d_2|n} f(d_1 d_2) = \left(\sum_{d_1|m} f(d_1)\right)\left(\sum_{d_2|n} f(d_2)\right) = g(m)g(n), \quad \text{v.s.v.}$$

(ii) We know (Chinese Remainder Theorem) that the function $\phi(n)$ satisfies (8). Hence, by part **(i)**, so also does the function $g(n) = \sum_{d|n} \phi(d)$. But, clearly, the function $h(n) = n$ also satisfies (8). We want to show that $g(n) = h(n)$ for all n. Since both satisfy (8), it suffices to prove this when n is a prime power, say $n = p^k$. But then

$$\sum_{d|p^k} \phi(d) = \sum_{i=0}^{k} \phi(p^i) = 1 + \sum_{i=1}^{k} (p^i - p^{i-1}) = p^k, \quad \text{v.s.v.}$$