The exponential generating function

Let $(u_n)_{n=0}^{\infty}$ be a sequence of numbers. The *exponential generating function* of the sequence is the power series

$$E(x) = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!}.$$

(i) Suppose the sequence (u_n) satisfies the recurrence relation

$$au_{n+2} + bu_{n+1} + cu_n = f_n,$$

where (f_n) is some 'known' sequence. Show that in that case the e.g.f. satisfies the differential equation

$$aE''(x) + bE'(x) + cE(x) = f(x),$$

where

$$f(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}.$$

We have thereby established an explicit correspondence between linear recurrence relations with constant coefficients and differential equations of the same type).

Exercises

Solve the following recurrence relations with the help of the exp.. gen. function (or otherwise)

1.

$$\begin{aligned} u_0 = 1, \quad u_1 = 2, \\ 2u_{n+2} = 7u_{n+1} - 3u_n + n, \ \ \forall\, n \geq 0. \end{aligned}$$

2.

$$\begin{aligned} u_0 = 4, \quad u_1 = 1, \\ u_n = 4u_{n-1} + 5u_{n-2} + 3^n, \ \ \forall\, n \geq 2. \end{aligned}$$

3.

$$\begin{aligned} u_0 = 3, \quad u_1 = 4, \\ u_{n+2} = 5u_{n+1} - 6u_n + n, \ \ \forall\, n \geq 0. \end{aligned}$$