Recurrence relations

Solve each of the following recurrence relations :

1. $a_0 = 1, a_{n+1} - a_n = 2n + 3 \; \forall \, n \geq 0.$
2. $a_0 = 3, a_{n+1} - a_n = 3n^2 - n, \; \forall \, n \geq 0.$
3. $a_0 = 1, a_{n+1} - 2a_n = 5, \; \forall \, n \geq 0.$
4. $a_0 = 1, a_{n+1} - 2a_n = 2^n, \; \forall \, n \geq 0.$
5. $a_0 = 0, a_1 = 1, a_{n+2} + 3a_{n+1} + 2a_n = 3^n, \; \forall \, n \geq 0.$
6. $a_0 = 1, a_1 = 2, a_{n+2} + 4a_{n+1} + 4a_n = 7, \; \forall \, n \geq 0.$
7. $a_0 = 1, a_1 = 2, a_{n+2} - 8a_{n+1} + 16a_n = 4^n, \; \forall \, n \geq 0.$

Answers :

1. $a_n = (n+1)^2.$
2. $a_n = 3 + n(n-1)^2.$
3. $a_n = 6 \cdot 2^n - 5.$
4. $a_n = 2^n + n \cdot 2^{n-1}.$
5.

$$a_n = \frac{3}{4} \cdot (-1)^n - \frac{4}{5} \cdot (-2)^n + \frac{1}{20} \cdot 3^n.$$

6.

$$a_n = \left(\frac{2}{9} - \frac{5n}{6} \right) \cdot (-2)^n + \frac{7}{9}.$$

7.

$$a_n = \left(1 - \frac{17n}{32} + \frac{n^2}{32} \right) \cdot 4^n.$$

Solutions to exercises from Grimaldi

SECTION 4.3 :

1 (e) If $a|x$ and $a|y$, then $x = ac$ and $y = ad$ for some $c, d \in \mathbf{Z}$. So $z = x - y = a(c-d)$, and $a|z$. The proofs for the other cases are similar.
(g) Follows from part **(f)** by an induction argument.

3. Since q is prime, its only positive divisors are 1 and q. With p a prime, it follows that $p > 1$. Hence $p|q \Rightarrow p = q$.
5. Suppose that $a|b$ or $a|c$. If $a|b$ then $ak = b$ for some $k \in \mathbf{Z}$. But $ak = b \Rightarrow (ak)c = a(kc) = bc \Rightarrow a|bc$. A similar result is obtained if $a|c$.
7 (a) Let $a = 1, b = 5, c = 2$. Another example is $a = b = 5, c = 3$.
(b) $31|(5a + 7b + 11c) \Rightarrow 31|(10a + 14b + 22c)$. Also, $31|(31a + 31b + 31c)$, so $31|[(31a + 31b + 31c) - (10a + 14b + 22c)]$, i.e.: $31|(21a + 17b + 9c)$, v.s.v..

9. $b|a$ and $b|(a + 2) \Rightarrow b|[(a + 2) - a]$, i.e.: $b|2$, so $b = 1$ or $b = 2$.
11. If x is odd then $x \equiv \pm 1 \pmod 4$. Thus $x^2 \equiv (\pm 1)^2 \equiv 1 \pmod 4$. Thus $a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod 4$, hence is not divisible by 4.
13. $7 \equiv 4 \pmod 3$.

Note : Exercises 14-22 not interesting for this course.

SECTION 4.5 :

7 (a) 96 **(b)** 270 **(c)** 144.
9. 660.
11. There are 252 possible values for n.
13 (i) The result is true. *Proof* : $10|a^2 \Leftrightarrow 2|a^2$ and $5|a^2 \Leftrightarrow 2|a$ and $5|a$, since 2 and 5 are primes, $\Leftrightarrow 10|a$.
(ii) False. Take $a = 2$ for example.

15. 176,400.
17. $n = 2 \cdot 3 \cdot 5^2 \cdot 7^2 = 7350$.
19 (a) 5 **(b)** 7 **(c)** 32 **(d)** $7 + 7 + 5 + 25 + 20 + 20 = 84$ **(e)** 84.
21. $1061 (= 512 + 256 + 293)$.
23 (a) $2^3 - 1 = 7$ such factorisations.

(b) $2^4 - 1 = 15$ such factorisations.
(c) If $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, then there are $2^{k-1} - 1$ such factorisations.

25. Grimaldi gives a proof by induction on n. I omit details.
27 (a) $56 = 1+2+4+7+14+28$ and $992 = 1+2+4+8+16+31+62+124+248+496$.
(c) This is a famous result of Euler. It follows from FTA that the divisors of $2^{m-1}(2^m - 1)$, when $2^m - 1$ is prime, are

$$1, 2, 2^2, 2^3, ..., 2^{m-1}, (2^m - 1), 2(2^m - 1), 2^2(2^m - 1), 2^3(2^m - 1), ..., 2^{m-1}(2^m - 1).$$

Now add 'em up !

SECTION 8.1 :

3 (a) 12 **(b)** 3.
5 (a) 534 **(b)** 458 **(c)** 76.
7. 4, 460, 400.
9.

$$\binom{37}{31} - \binom{7}{1}\binom{27}{21} + \binom{7}{2}\binom{17}{11} - \binom{7}{3}\binom{7}{1}.$$

11.

$$(15!) \cdot \left[\binom{14}{10} - \binom{5}{1}\binom{10}{6} + \binom{5}{2}\binom{6}{2}\right].$$

13. $26! - [3(23!) + 24!] + (20! + 21!)$.
15.

$$\frac{1}{6^8} \cdot \left[6^8 - \binom{6}{1}5^8 + \binom{6}{2}4^8 - \binom{6}{3}3^8 + \binom{6}{4}2^8 - \binom{6}{5}\right].$$

17.

$$\frac{9!}{[(3!)^3]} - 3\left[\frac{7!}{(3!)^2}\right] + 3\left(\frac{5!}{3!}\right) - 3!$$

19. $651/7776 \approx 0.08372$.
21 (a) 32 **(b)** 96 **(c)** 3200.
23 (a) 2^{n-1} **(b)** $2^{n-1}(p-1)$.

25 (a) 1600 **(b)** 4399.
27. $\phi(17) = \phi(32) = \phi(48) = 16$.
29. If 4 divides $\phi(n)$ then one of the following must hold :
(i) n is divisible by 8,
(ii) n is divisible by two (or more) distinct odd primes,
(iii) n is divisible by an odd prime $p \equiv 1 \pmod 4$,
(iv) n is divisible by 4 and at least one odd prime, and not divisible by 8.

SECTION 14.5 :

1 (a) Yes, No, Yes **(b)** No, Yes, Yes.
3 (a) -6,1,8,15 **(b)** -9,2,13,24 **(c)** -7,10,27,44.
5. $a \equiv b \pmod n \Leftrightarrow n|a-b \Rightarrow m|a-b \Leftrightarrow a \equiv b \pmod m$.
7. For example, $a = 8$, $b = 2$, $m = 6$, $n = 2$.
9. The sum is $n(n-1)/2$, as may be proven by several different means. If n is odd, then $(n-1)/2$ is an integer, so the sum is a multiple of n, v.s.v. If n is even, then $n/2$ is an integer, and we note that $n(n-1)/2 = (\frac{n}{2}-1)\cdot n + \frac{n}{2}$, which proves that the sum is congruent to $\frac{n}{2} \pmod n$.
11 (b) No, for example $2\mathcal{R}3$ and $3\mathcal{R}5$ but $5 \not\mathcal{R}8$. Another example is $2\mathcal{R}3$ and $2\mathcal{R}5$ but $4 \not\mathcal{R}15$.
13 (a) $(17)^{-1} \equiv 831$ **(b)** $(100)^{-1} \equiv 111$ **(c)** $(777)^{-1} \equiv 735$.
15 (a) 16 units, 0 proper zero divisors.
(b) 72 units, 44 proper zero divisors.
(c) 1116 units, 0 proper zero divisors.

17.

$$\frac{1}{\binom{1000}{3}} \cdot \left[\binom{334}{3} + 2\binom{333}{3} + \binom{334}{1}\binom{333}{1}^2 \right]$$

19. We discussed this at lektioner.
21. $a \equiv b \pmod n \Leftrightarrow a = b + kn$ for some $k \in \mathbf{Z}$. Now $d|b$ and $d|n \Leftrightarrow d|b+kn$ and $d|n \Leftrightarrow d|a$ and $d|n$. From which it follows that $\mathrm{GCD}(a,n) = \mathrm{GCD}(b,n)$.
23. DOOJDXOLVGLYLGHGLQWRWKUHHSDUWV